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Wheeler and Feynman electrodynamics within the framework of retarded causality

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Abstract
A frontal collision of two point-like charged particles which are asymptotically free in the remote past and in the distant future is considered. Ten conserved quantities corresponding to the symmetry of a closed system of particles and electromagnetic field under the Poincaré group are expressed in terms of particle variables. It is shown that an interference of outgoing electromagnetic waves (retarded Liénard–Wiechert solutions) ensures the action of the field of one source on another (mutual interaction). The combination of wave motions accords with the modified Wheeler and Feynman *absorber theory of radiation* where (acausal) 'perfect absorption' is replaced by an interference phenomenon.

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1. Introduction

In classical field theory particles interact with one another through the medium of a field which has its own uncountable infinite degrees of freedom. Roughly speaking, the set of equations of motion is divided into two subsets: (i) one defines the evolution of field variables; (ii) the other describes the behaviour of the particles.

One can try to solve the first subset in order to express the field variables in terms of particle variables. These 'fields' do not have degrees of freedom of their own: they are functionals of particle paths. Substituting these *direct particle fields* [1] for 'true' fields in the second subset yields equations of motion in particle variables. This immediately implies *action-at-a-distance*: the particles interact with one another directly. Nevertheless, the delay of disturbances which are propagated with a finite velocity is taken into account. As discussed in [1], the concept of delayed action-at-a-distance was first described by Gauss in 1845.

Liénard–Wiechert fields are the solutions of Maxwell equations with point-like sources. In his classical paper [2], Dirac used them in the law of conservation of the total four-momentum of a composite (particle plus field) system. It provides the foundation for his derivation of the radiation–reaction force.

Teitelboim [3] divides the energy–momentum carried by a (retarded) Liénard–Wiechert field into a bound component and a radiative component. The bound part is permanently 'attached' to the charge and is carried along with it; the radiation part detaches itself from the charge and leads an independent existence (i.e. the integral of the Larmor relativistic rate of radiated energy–momentum over the particle's world line). In fact, a charged particle cannot be separated from its bound electromagnetic 'cloud', so that the four-momentum of the particle is the sum of the mechanical momentum and the electromagnetic bound four-momentum.

The author [4] has found recently ten conserved quantities corresponding to the symmetry of the composite system of point-like charged particle and its own electromagnetic field under Poincaré group. (It was assumed that the particle moves arbitrarily.) Similarly to the energy–momentum, the angular momentum and centre-of-mass conserved quantity decompose naturally into a particle component and a radiative component. The former depends on the instant characteristic of the charged particle while the latter accumulates with time. (The angular momentum arises from the invariance of the system under space rotations while the centre-of-mass conserved quantity is due to invariance under Lorentz transformation, see [5].)

The Liénard–Wiechert field has a singularity at the position of a particle. A divergent self-energy term arises unavoidably whenever one introduces a point charge in classical electrodynamics. Teitelboim shows [3] that it cannot be separated from the four-momentum of a charged particle. The angular momentum and centre-of-mass conserved quantity contain the (divergent) four-momentum in the proper place [4]. To reconcile the theory with observation, an additional 'renormalization assumption' is necessary. It is sufficient to introduce *finite* four-momenta of the charged particles as those which possess the true physical meaning.

Conserved quantities place stringent requirements on the dynamics of the system. They demand that the change in electromagnetic field momentum and total angular momentum should be balanced by a corresponding change in the momentum and total angular momentum of the particle, so that the total four-momentum $(p^0, \boldsymbol{p})$ and angular momentum $(\boldsymbol{J}, \boldsymbol{K})$ are properly conserved.

To construct the particle equation of motion we only need to consider the vicinity of the world line at a fixed instant of time. Derivatives of the angular momentum and centre-of-mass conserved quantity constitute a system of six linear equations in four particle momentum components (see [4], equations (3.10) and (3.11)). It was pointed out that Teitelboim's expression [3] for the four-momentum of accelerated point-like charge,

$$p^{\mu}_{\text{part}} = mu^{\mu} - \tfrac{2}{3}e^2a^{\mu} \tag{1.1}$$

is inconsistent with the structure of the derivative of the centre-of-mass conserved quantity. (Teitelboim's expression agrees with the Lorentz–Dirac equation [2] of motion of a charged particle under the influence of an external force as well as its own electromagnetic field.) Moreover, the system does not possess a solution whenever a particle's motion is accelerated (if not the usual velocity term mu^{μ} satisfies this system). Therefore, if the particle is not acted upon by an external force, the motion satisfies the law of inertia (Newton's first law). The problem of runaway solutions (where acceleration increases exponentially with time) does not occur. The question is what expression should be used instead of (1.1) to describe the four-momentum of a point-like charge in the presence of an external device?

In Rohrlich's opinion, the answer cannot be found in the usual analysis of a heuristic model. In ([5], section 6.2) Rohrlich states that the object is 'to find a formulation of classical charged particle theory which does not require any reference to, or assumptions about, the particle structure, its charge distribution and its size'. A more fruitful clue is the investigation of conserved quantities, since they place stringent requirements on the particle's behaviour. In this paper, we shall determine the ten conserved quantities corresponding to Poincaré symmetry of a closed system of two point-like charged particles and their electromagnetic fields. To simplify the problem as much as possible we consider a typical scattering event: a frontal collision of two asymptotically free charged particles. The words 'asymptotically free' mean the asymptotic conditions defined in ([5], section 6.4).

An external device will be modelled by a very massive particle (i.e. one very massive charge, and one light one).

The so-called *direct particle fields* [1] will be used in the calculation of the energy–momentum and total angular momentum carried by the 'two-particle' electromagnetic field. They are derived from Liénard–Wiechert potentials as solutions of Maxwell equations for arbitrarily moving point-like sources. Therefore, we work within the realm of *action-at-a-distance* electrodynamics [1].

The theory was elaborated by Wheeler and Feynman [6]. It is based on the following assumptions ([6], p 160):

(1) 'An accelerated point charge in otherwise charge-free space does not radiate electromagnetic energy.
(2) The fields which act on a given particle arise only from other particles.
(3) These fields are represented by one-half the retarded plus one-half the advanced Liénard–Wiechert solutions of Maxwell's equations. This law is symmetric with respect to past and future.
(4) Sufficiently many particles are present to absorb completely the radiation given off by the source'.

Since the source emanates in all possible directions, all the particles of the universe are required to absorb the radiation completely. They constitute a *perfect absorber* which possesses a remarkable twofold property: it cancels the (acausal) advanced part of the fields acting on a given particle and doubles the retarded one. Therefore, the complete absorption is the crucial issue of the theory. For this reason Wheeler and Feynman called it the *absorber theory of radiation*.

Rigorous calculations performed in the present paper reveal that the combination of retarded Liénard–Wiechert fields forms a resultant field with the desired properties. Then the 'perfect absorption' should be replaced by the interference of outgoing waves in Wheeler and Feynman electrodynamics. It allows us to reconcile the Wheeler and Feynman theory with the concept of retarded causality.

2. Preliminaries

We choose metric tensor $\eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$ for Minkowski space $\mathbb{M}_4$. We use the Heaviside–Lorentz system of units with the velocity of light $c = 1$. Summation over repeated indices is understood throughout the paper; Greek indices run from 0 to 3, and Latin indices from 1 to 3. The particles' coordinate, velocity, etc are labelled a or b.

We consider a typical scattering event where two particles are asymptotically free in the remote past and in the distant future (see [5], section 6.4). Average velocities are not large enough to initiate particle creation and annihilation.

We suppose that the particles move along the z-axis. Their world lines

$$\zeta_a : \mathbb{R} \to \mathbb{M}_4 \qquad t \mapsto (t, 0, 0, z_a(t)) \tag{2.1}$$

are meant as local sections of trivial bundle $(\mathbb{M}_4, i, \mathbb{R})$ where the projection

$$i : \mathbb{M}_4 \to \mathbb{R} \qquad (y^0, y^i) \mapsto y^0 \tag{2.2}$$

defines the instant form of dynamics [7].

Having stated our notation we now write the components of total four-momentum of a closed system of particles and electromagnetic field as follows:

$$p^\nu(t) = p^\nu_{\rm mech}(t) + \int_{\Sigma_t} {\rm d}\sigma_\mu T^{\mu\nu}. \tag{2.3}$$

The first term is a sum of mechanical momenta of the particles while the volume integral defines the momentum four-vector of the electromagnetic field. We define ${\rm d}\sigma_\mu$ the vectorial surface element on a spacelike hypersurface Σ_t which intersects world lines ζ_1 and ζ_2 at the points $(t, 0, 0, z_1(t))$ and $(t, 0, 0, z_2(t))$, respectively. (By Σ_t we take a fibre [8] of 'instant' bundle (2.2) over $t \in \mathbb{R}$.) By $T^{\mu\nu}$ we denote the components of the Maxwell energy–momentum tensor density

$$4\pi T^{\mu\nu} = f^{\mu\lambda} f^\nu{}_\lambda - 1/4\eta^{\mu\nu} f^{\kappa\lambda} f_{\kappa\lambda} \tag{2.4}$$

where field strengths $f^{\mu\lambda}$ are the sum of direct particle fields $f^{\mu\lambda}_{(1)}$ and $f^{\mu\lambda}_{(2)}$ associated with the first and second particles, respectively. (The retarded Liénard–Wiechert solutions are meant.) So, the total electromagnetic field stress–energy tensor (2.4) is

$$T^{\mu\nu} = T^{\mu\nu}_{(1)} + T^{\mu\nu}_{(2)} + T^{\mu\nu}_{\rm int}. \tag{2.5}$$

The $T^{\mu\nu}_{(a)}$ term is given by the expression (2.4) where 'total' field strengths $f^{\mu\lambda}$ are replaced by 'individual' ones $f^{\mu\lambda}_{(a)}$. The interference term

$$4\pi T^{\mu\nu}_{\rm int} = f^{\mu\lambda}_{(1)} f^\nu_{(2)\lambda} + f^{\mu\lambda}_{(2)} f^\nu_{(1)\lambda} - 1/4\eta^{\mu\nu}\left(f^{\kappa\lambda}_{(1)} f^{(2)}_{\kappa\lambda} + f^{\kappa\lambda}_{(2)} f^{(1)}_{\kappa\lambda}\right) \tag{2.6}$$

describes the combination of electromagnetic fields.

3. 'Interference' coordinate system

The volume integration of 'one-particle' energy–momentum tensor density is performed in [4]. It is shown that the computation of the electromagnetic field momentum which flows across hyperplane $\Sigma_t = \{y \in \mathbb{M}_4 : y^0 = t\}$ at a fixed instant of time t does not contradict the corresponding calculation [2, 3, 5] performed in a manifestly covariant way. An appropriate coordinate system for flat spacetime is used in [4]. It is a specific example of the Newman and Unti [9] class of coordinate systems centred on an (accelerated) world line $\zeta : \mathbb{R} \to \mathbb{M}_4$ of the particle. Minkowski space $\mathbb{M}_4$ becomes a disjoint union of fibres $i^{-1}(t) := \Sigma_t$ of the trivial bundle (2.2). A fibre Σ_t is a disjoint union of (retarded) spheres centred on a world line of the particle. The sphere

$$S(z(u), t-u) = \left\{ y \in \mathbb{M}_4 : (y^0 - u)^2 = \sum_i (y^i - z^i(u))^2, y^0 = t, t-u > 0 \right\}$$

is the intersection of the future light cone, generated by null rays emanating from $z(u) \in \zeta$ in all possible directions, and the hyperplane Σ_t. For the fixed instant t the retarded time parameter $u \in]-\infty, t] \subset \mathbb{R}$. Points on the sphere are distinguished by spherical polar angles.

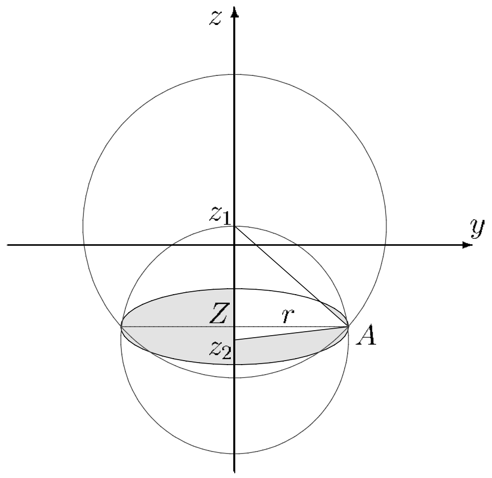

Figure 1. The sphere $S_1(z_1(t_1), t - t_1)$ is the intersection of the future light cone with vertex at $(t_1, 0, 0, z_1) \in \zeta_1$ and hyperplane Σ_t. The sphere $S_2(z_2(t_2), t - t_2)$ is the intersection of Σ_t and the forward light cone of $(t_2, 0, 0, z_2) \in \zeta_2$. The support of the integral of the interference term (2.6) is the circle $C(Z, r) = S_1 \cap S_2$.

The main goal of the present paper is to find the components of total four-momentum carried by the electromagnetic field of two charged particles. Volume integration of the first term of the 'two-particle' stress–energy tensor (2.5) can be handled via the coordinate system centred on the first world line. To calculate the integral of the second term of tensor (2.5) we use the coordinate system centred on the second world line. Another coordinate system is necessary to compute the interference part of the energy and momentum carried by radiation.

It is suitable to consider the situation from a geometrical point of view as shown in figure 1. To sum up the product of the fields produced at given instants t_1 and t_2, we must restrict the integral to the intersection of the sphere $S(z_1(t_1), t - t_1)$ and the sphere $S(z_2(t_2), t - t_2)$. It is the circle $C(Z, r)$ centred at the point

$$Z(t, t_1, t_2) = \frac{1}{2}[z_1(t_1) + z_2(t_2)] + \frac{(t_1 - t_2)(2t - t_1 - t_2)}{2[z_1(t_1) - z_2(t_2)]}. \tag{3.1}$$

(We assume that $z_1(t_1) > z_2(t_2)$ for all values of the retarded times $t_1 \in]-\infty, t]$ and $t_2 \in]-\infty, t]$.) The square of the radius r of the circle can be expressed in the following alternative ways:

$$\begin{aligned} r^2 &= (t - t_1)^2 - (Z - z_1)^2 \\ &= (t - t_2)^2 - (Z - z_2)^2 \\ &= \tfrac{1}{2}[(t - t_1)^2 + (t - t_2)^2 - q^2] - (Z - z_1)(Z - z_2). \end{aligned} \tag{3.2}$$

The characteristics of the circle are obtained from the analysis of the triangle $z_1 z_2 A$ with sides $|z_1 A| = t - t_1$, $|z_2 A| = t - t_2$ and $|z_1 z_2| = z_1(t_1) - z_2(t_2) := q$.

Figure 1 shows that one-to-one correspondence $(y^\alpha) \mapsto (t, t_1, t_2, \varphi)$ looks like a transformation to well-known cylindrical coordinates

$$y^0 = t \qquad y^1 = r(t, t_1, t_2) \sin\varphi \qquad y^2 = r(t, t_1, t_2) \cos\varphi \qquad y^3 = Z(t, t_1, t_2). \tag{3.3}$$

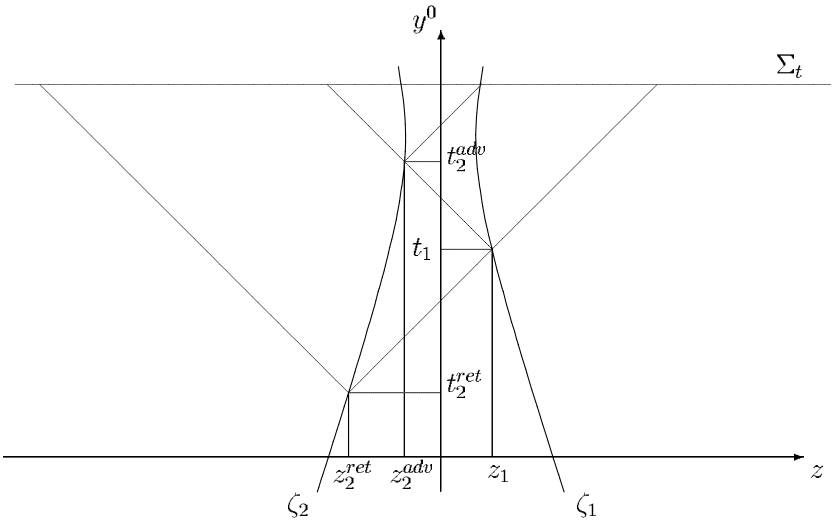

Figure 2. For a given t_1 the retarded time t_2 increases from $t_2^{\mathrm{ret}}(t_1)$ to $t_2^{\mathrm{adv}}(t_1)$. Minimal value $t_2^{\mathrm{ret}}(t_1)$ labels the vertex of the forward light cone which is punctured by the world line of the first charge at a given point $(t_1, 0, 0, z_1(t_1))$. The world line of the second charge punctures the future light cone of this point at $(t_2^{\mathrm{adv}}(t_1), 0, 0, z_2^{\mathrm{adv}})$.

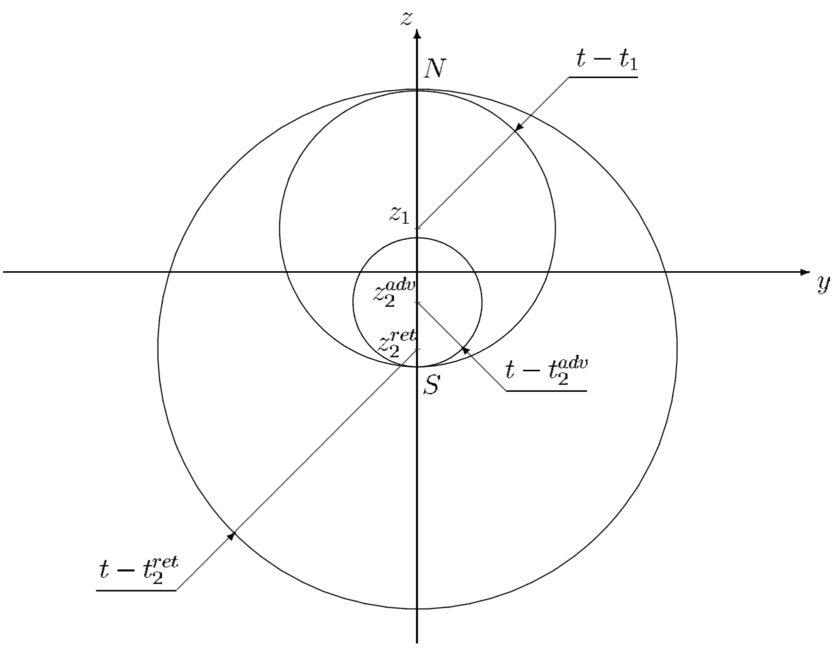

Figure 3. The sphere $S_2(z_2^{\mathrm{ret}}, t - t_2^{\mathrm{ret}})$ is the intersection of the future light cone at $(t_2^{\mathrm{ret}}, 0, 0, z_2^{\mathrm{ret}})$ and Σ_t. It touches a given sphere $S_1(z_1, t - t_1)$ at point N. The sphere $S_2(z_2^{\mathrm{adv}}, t - t_2^{\mathrm{adv}})$ touches $S_1(z_1, t - t_1)$ at point S. If retarded time t_2 increases from $t_2^{\mathrm{ret}}(t_1)$ to $t_2^{\mathrm{adv}}(t_1)$ the sphere S_1 is covered by circles $C(Z, r) = S_1 \cap S_2$.

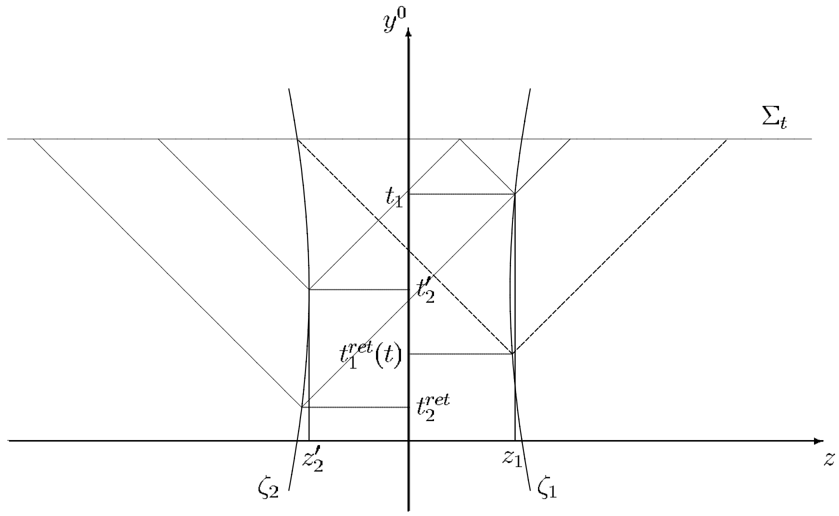

Figure 4. The forward light cone of $(t_1^{\rm ret}(t), 0, 0, z_1^{\rm ret})$ touches the second world line at the instant of observation. Future light cones of upper vertices do not intersect it at all. For a given $t_1 \in [t_1^{\rm ret}(t), t]$ the parameter t_2 increases from $t_2^{\rm ret}(t_1)$ to $t_2'(t, t_1)$. The maximal value $t_2'(t, t_1)$ labels the vertex of future light cone which touches the forward light cone of $(t_1, 0, 0, z_1)$. The minimal value of t_2 is the solution $t_2^{\rm ret}(t_1)$ of equation (3.4).

To cover the sphere $S_1(z_1(t_1), t - t_1)$ where t_1 is fixed we change the parameter t_2. The starting point is the solution $t_2^{\rm ret}(t_1)$ of the algebraic equation

$$t_1 - t_2^{\rm ret} = z_1(t_1) - z_2\big(t_2^{\rm ret}\big) \tag{3.4}$$

which describes the future light cone with vertex at $\big(t_2^{\rm ret}, 0, 0, z_2^{\rm ret}\big)$ (see figure 2). The sphere $S_2\big(z_2^{\rm ret}, t - t_2^{\rm ret}\big)$ touches a given sphere $S_1(z_1(t_1), t - t_1)$ at the North pole (see figure 3). If parameter t_2 increases to $t_2^{\rm adv}(t_1)$ being the solution of the algebraic equation

$$t_2^{\rm adv} - t_1 = z_1(t_1) - z_2\big(t_2^{\rm adv}\big) \tag{3.5}$$

we arrive at the South pole of the sphere S_1. Equation (3.5) looks like the equation of the *backward* light cone of $\big(t_2^{\rm adv}, 0, 0, z_2^{\rm adv}\big)$, but it defines the *future* light cone with vertex at $(t_1, 0, 0, z_1)$ (see figure 2). The sphere S_1 becomes the disjoint union of circles $C(Z, r) = S_1 \cap S_2$ if the parameter t_2 changes from $t_2^{\rm ret}(t_1)$ to $t_2^{\rm adv}(t_1)$.

Going along the world line of the first charge we arrive unavoidably at the point $t_1^{\rm ret}(t)$ being the solution of the algebraic equation

$$t - t_1^{\rm ret} = z_1\big(t_1^{\rm ret}\big) - z_2(t). \tag{3.6}$$

The forward light cone of this point touches the world line of the second charge at point $(t, 0, 0, z_2(t))$ (see figure 4). Light cones of upper vertices do not intersect the second world line at all. Spheres $S_1(z_1(t_1), t - t_1)$ determined by $t_1 \in \big[t_1^{\rm ret}(t), t\big]$ constitute the region of hyperplane Σ_t which requires another parametrization. For a given instant t_1 from this interval the South pole S (see figure 5) is associated with the solution $t_2'(t_1)$ of the following equation:

$$2t - t_1 - t_2' = z_1(t_1) - z_2(t_2'). \tag{3.7}$$

The North pole N is still connected with the solution $t_2^{\rm ret}(t_1)$ of equation (3.4).

So, we construct the global coordinate system centred on the world line of the first particle. It is based on the trivial fibre bundle (2.2). A fibre Σ_t is a disjoint union of retarded spheres

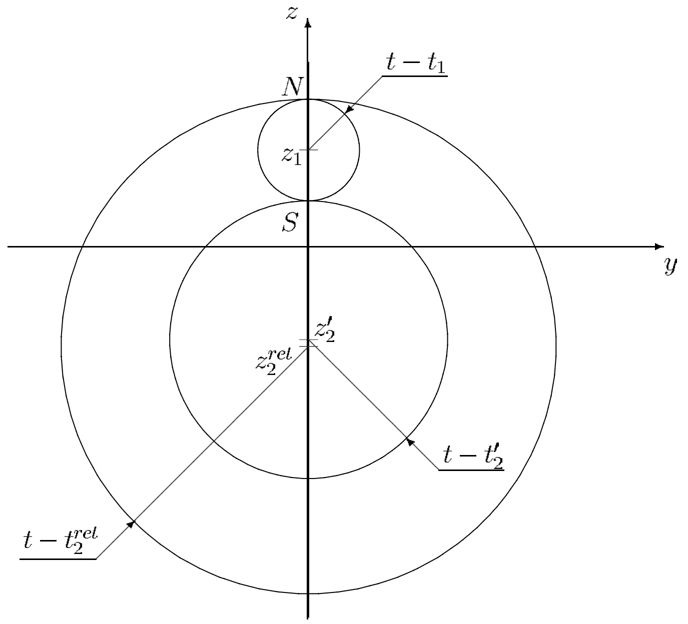

Figure 5. For a given $t_1 \in [t_1^{\rm ret}(t), t]$ the sphere $S_1(z_1, t - t_1)$ is a disjoint union of circles $C(Z, r) = S_1 \cap S_2$. Their radius r and centre coordinate Z are determined by t_2. The parameter t_2 increases from $t_2^{\rm ret}(t_1)$ (North pole) to $t_2'(t, t_1)$ (South pole); $\varphi \in [0, 2\pi]$.

S_1 centred on the world line of the first particle. A sphere is parametrized by the retarded time of the second particle and the polar angle. Locally the coordinate transformation is given by equations (3.3).

In an analogous way we construct the coordinate system centred on the world line of the second particle. If $t_2 \in \,]-\infty, t_2^{\rm ret}(t)]$ then $t_1 \in \left[t_1^{\rm ret}(t_2), t_1^{\rm adv}(t_2)\right]$; if $t_2 \in \left[t_2^{\rm ret}(t), t\right]$ then $t_1 \in \left[t_1^{\rm ret}(t_2), t_1'(t, t_2)\right]$, $\varphi \in [0, 2\pi[$. The ends of the intervals are defined by the following algebraic equations:

$$t_2 - t_1^{\rm ret} = z_1\left(t_1^{\rm ret}\right) - z_2(t_2) \tag{3.8}$$

$$t_1^{\rm adv} - t_2 = z_1\left(t_1^{\rm adv}\right) - z_2(t_2) \tag{3.9}$$

$$t - t_2^{\rm ret} = z_1(t) - z_2\left(t_2^{\rm ret}\right) \tag{3.10}$$

$$2t - t_1' - t_2 = z_1(t_1') - z_2(t_2). \tag{3.11}$$

It is worth noting that the functions $t_1^{\rm ret}(t_2)$ and $t_2^{\rm adv}(t_1)$ are inverted with respect to each other as well as the pair of functions $t_1^{\rm adv}(t_2)$ and $t_2^{\rm ret}(t_1)$. For a fixed observation time t the functions $t_1'(t, t_2)$ and $t_2'(t, t_1)$ are inverses too.

4. Electromagnetic fields in terms of 'interference' coordinates

The components of Liénard–Wiechert potentials $\hat{A}^{(a)} = A_\alpha^{(a)}\, {\rm d}y^\alpha$ depend on the state of the particle motion at the retarded time t_a only:

$$A_\alpha^{(a)} = e_a \frac{u_\alpha(t_a)}{R_a(y)}. \tag{4.1}$$

Here $u_\alpha(t_a)$ are the components of velocity one-form $\hat{u}^{(a)}$ and R_a is the so-called retarded distance [10]:

$$R_a(y) = -\eta_{\alpha\beta}(y^\alpha - z^\alpha(t_a))u^\beta(t_a). \tag{4.2}$$

Inserting (3.3) into (4.2) and (4.1) and taking into account (2.1), we obtain

$$A_0^{(a)} = -\frac{e_a}{r_a} \qquad A_1^{(a)} = 0 \qquad A_2^{(a)} = 0 \qquad A_3^{(a)} = e_a\frac{v_a}{r_a} \tag{4.3}$$

where $r_a = t - t_a - (Z - z_a)v_a$, $v_a \equiv \mathrm{d}z_a(t_a)/\mathrm{d}t_a$.

The direct particle field is defined in terms of the four-potential (4.1) by $f_{\alpha\beta}^{(a)} = A_{\beta,\alpha}^{(a)} - A_{\alpha,\beta}^{(a)}$. Differentiation of coordinate transformation (3.3) yields

$$\begin{aligned}
\frac{\partial}{\partial y^0} &= \frac{\partial}{\partial t} + \frac{t-t_1}{r_1}\frac{\partial}{\partial t_1} + \frac{t-t_2}{r_2}\frac{\partial}{\partial t_2} \\
\frac{\partial}{\partial y^1} &= -r\sin\varphi\left(\frac{1}{r_1}\frac{\partial}{\partial t_1} + \frac{1}{r_2}\frac{\partial}{\partial t_2}\right) + \frac{\cos\varphi}{r}\frac{\partial}{\partial\varphi} \\
\frac{\partial}{\partial y^2} &= -r\cos\varphi\left(\frac{1}{r_1}\frac{\partial}{\partial t_1} + \frac{1}{r_2}\frac{\partial}{\partial t_2}\right) - \frac{\sin\varphi}{r}\frac{\partial}{\partial\varphi} \\
\frac{\partial}{\partial y^3} &= -\frac{Z-z_1}{r_1}\frac{\partial}{\partial t_1} - \frac{Z-z_2}{r_2}\frac{\partial}{\partial t_2}.
\end{aligned} \tag{4.4}$$

It is easy to check that the retarded time derivatives of (non-trivial) z-coordinate (3.1) of the centre are as follows:

$$\frac{\partial Z}{\partial t_1} = \frac{r_1}{q} \qquad \frac{\partial Z}{\partial t_2} = -\frac{r_2}{q}. \tag{4.5}$$

Having used the differential rules (4.4), we express the direct field components in terms of 'interference' coordinates as follows:

$$\begin{aligned}
f_{01}^{(a)} &= -\frac{e_a}{r_a^3}r\sin\varphi\, a_a \qquad f_{02}^{(a)} = -\frac{e_a}{r_a^3}r\cos\varphi\, a_a \\
f_{03}^{(a)} &= \frac{e_a}{r_a^3}[(t-t_a)b_a - (Z-z_a)a_a] \\
&= \frac{e_a}{r_a^3}[v_a(t-t_a) - (Z-z_a)]\left(1-v_a^2\right) + \frac{e_a}{r_a^3}r^2\dot{v}_a \\
f_{12}^{(a)} &= 0 \qquad f_{13}^{(a)} = -\frac{e_a}{r_a^3}r\sin\varphi\, b_a \qquad f_{23}^{(a)} = -\frac{e_a}{r_a^3}r\cos\varphi\, b_a.
\end{aligned} \tag{4.6}$$

In the above expressions

$$a_a = 1 - v_a^2 + (Z-z_a)\dot{v}_a \qquad b_a = v_a\left(1-v_a^2\right) + (t-t_a)\dot{v}_a \tag{4.7}$$

where $\dot{v}_a \equiv \mathrm{d}v_a(t_a)/\mathrm{d}t_a$.

5. Interference part of the electromagnetic field four-momentum

Now, we calculate the interference part of the energy and momentum carried by the 'two-particle' electromagnetic field:

$$p_{\mathrm{int}}^\mu(t) = \int_{\Sigma_t} \mathrm{d}\sigma_0\, T_{\mathrm{int}}^{0\mu}. \tag{5.1}$$

An integration hypersurface $\Sigma_t = \{y \in \mathbb{M}_4 : y^0 = t\}$ is a surface of constant t. The surface element is given by $\mathrm{d}\sigma_0 = \sqrt{-g}\,\mathrm{d}t_1\,\mathrm{d}t_2\,\mathrm{d}\varphi$ where

$$\sqrt{-g} = \frac{r_1 r_2}{q} \tag{5.2}$$

is the determinant of the metric tensor of Minkowski space viewed in curvilinear coordinates (3.3).

5.1. Interference part of space components

It is straightforward to substitute the components (4.6) into equation (2.6) to calculate the interference part of the electromagnetic field stress–energy tensor. We obtain the following momentum densities:

$$\begin{aligned}
4\pi T^{01}_{\text{int}} &= \frac{e_1 e_2}{(r_1 r_2)^3} r \sin\varphi[(2t - t_1 - t_2)b_1 b_2 - (Z - z_1)a_1 b_2 - (Z - z_2)a_2 b_1]\\
4\pi T^{02}_{\text{int}} &= \frac{e_1 e_2}{(r_1 r_2)^3} r \cos\varphi[(2t - t_1 - t_2)b_1 b_2 - (Z - z_1)a_1 b_2 - (Z - z_2)a_2 b_1]\\
4\pi T^{03}_{\text{int}} &= \frac{e_1 e_2}{(r_1 r_2)^3} r^2[a_1 b_2 + a_2 b_1].
\end{aligned} \tag{5.3}$$

The volume integration (5.1) of the interference part of the 'emitted tensor' can be performed via the coordinate system centred on a world line either of the first particle

$$\left[\int_{-\infty}^{t_1^{\text{ret}}(t)} \mathrm{d}t_1 \int_{t_2^{\text{ret}}(t_1)}^{t_2^{\text{adv}}(t_1)} \mathrm{d}t_2 + \int_{t_1^{\text{ret}}(t)}^{t} \mathrm{d}t_1 \int_{t_2^{\text{ret}}(t_1)}^{t_2'(t,t_1)} \mathrm{d}t_2\right] \int_0^{2\pi} \mathrm{d}\varphi \frac{r_1 r_2}{q} \tag{5.4}$$

or of the second particle

$$\left[\int_{-\infty}^{t_2^{\text{ret}}(t)} \mathrm{d}t_2 \int_{t_1^{\text{ret}}(t_2)}^{t_1^{\text{adv}}(t_2)} \mathrm{d}t_1 + \int_{t_2^{\text{ret}}(t)}^{t} \mathrm{d}t_2 \int_{t_1^{\text{ret}}(t_2)}^{t_1'(t,t_2)} \mathrm{d}t_1\right] \int_0^{2\pi} \mathrm{d}\varphi \frac{r_1 r_2}{q}. \tag{5.5}$$

As the momentum density T^{01}_{int} is proportional to $\sin\varphi$, p^1_{int} vanishes due to angle integration. The component p^2_{int} is equal to zero because T^{02}_{int} is proportional to $\cos\varphi$. The third component p^3_{int} is non-trivial only.

Integrand $\sqrt{-g}\,T^{03}_{\text{int}}$ has the remarkable property of being the sum of two partial derivatives:

$$\begin{aligned}
\frac{1}{q(r_1 r_2)^2} r^2[a_1 b_2 + a_2 b_1] &= \frac{\partial}{\partial t_1}\left[r^2 \frac{b_2}{q r_1 r_2^2} - \frac{(t - t_1)^2\left(1 - v_1^2\right)}{q^2 r_1^2}\right]\\
&\quad + \frac{\partial}{\partial t_2}\left[r^2 \frac{b_1}{q r_1^2 r_2} + \frac{(t - t_2)^2\left(1 - v_2^2\right)}{q^2 r_2^2}\right].
\end{aligned} \tag{5.6}$$

(Prefactor $e_1 e_2/4\pi$ is omitted.) It is natural to integrate the first term according to the rule (5.5) and the second one according to the rule (5.4). The angle integration gives the factor 2π. The limits of 'inner' integrals are valuable only in the integration procedure. The square of radius (3.2) is equal to zero at the end points (3.4)–(3.11). Therefore, the terms which are proportional to r^2 vanish due to 'inner' integration.

The integral to be computed then takes the form

$$
\begin{aligned}
p^3_{\rm int} = \frac{e_1e_2}{2}\Bigg\{ &-\int_{-\infty}^{t_2^{\rm ret}(t)} \mathrm{d}t_2 \frac{1}{\left[z_1\left(t_1^{\rm adv}\right)-z_2(t_2)\right]^2}\frac{1+v_1\left(t_1^{\rm adv}\right)}{1-v_1\left(t_1^{\rm adv}\right)}\\
&+\int_{-\infty}^{t} \mathrm{d}t_2 \frac{1}{\left[z_1\left(t_1^{\rm ret}\right)-z_2(t_2)\right]^2}\frac{1-v_1\left(t_1^{\rm ret}\right)}{1+v_1\left(t_1^{\rm ret}\right)}\\
&-\int_{t_2^{\rm ret}(t)}^{t} \mathrm{d}t_2 \frac{1}{[z_1(t_1')-z_2(t_2)]^2}\frac{1-v_1(t_1')}{1+v_1(t_1')}\Bigg\}\\
&+\frac{e_1e_2}{2}\Bigg\{\int_{-\infty}^{t_1^{\rm ret}(t)} \mathrm{d}t_1 \frac{1}{\left[z_1(t_1)-z_2\left(t_2^{\rm adv}\right)\right]^2}\frac{1-v_2\left(t_2^{\rm adv}\right)}{1+v_2\left(t_2^{\rm adv}\right)}\\
&-\int_{-\infty}^{t} \mathrm{d}t_1 \frac{1}{\left[z_1(t_1)-z_2\left(t_2^{\rm ret}\right)\right]^2}\frac{1+v_2\left(t_2^{\rm ret}\right)}{1-v_2\left(t_2^{\rm ret}\right)}\\
&+\int_{t_1^{\rm ret}(t)}^{t} \mathrm{d}t_1 \frac{1}{[z_1(t_1)-z_2(t_2')]^2}\frac{1+v_2(t_2')}{1-v_2(t_2')}\Bigg\}.
\end{aligned}
\tag{5.7}
$$

To understand the situation more thoroughly, we analyse the expressions under the integral signs.

Let us consider the terms written in between the first braces. The third component of the Lorentz force acting on the second charge is found under the second integral sign. Indeed, expressions (4.6) prompt that

$$
\frac{e_1}{\left[z_1\left(t_1^{\rm ret}\right)-z_2(t_2)\right]^2}\frac{1-v_1\left(t_1^{\rm ret}\right)}{1+v_1\left(t_1^{\rm ret}\right)}
$$

is the third component $F_{03}^{(1)}(\text{ret})$ of the electric field strength produced by the first charge e_1 which is located at the point $\left(t_1^{\rm ret}, 0, 0, z_1^{\rm ret}\right)$. (The other components (4.6) of $\hat{f}^{(1)}$ vanish at points of the world line of the second particle where $r = 0$.) The integral of this Lorentz force over the path of the second particle can be interpreted as the work done by this force.

The first term contains the integrand

$$
-\frac{e_1}{\left[z_1\left(t_1^{\rm adv}\right)-z_2(t_2)\right]^2}\frac{1+v_1\left(t_1^{\rm adv}\right)}{1-v_1\left(t_1^{\rm adv}\right)}.
$$

It looks like the third component $F_{03}^{(1)}(\text{adv})$ of the electric field strength generated by the first charge which is located at the point $\left(t_1^{\rm adv}, 0, 0, z_1^{\rm adv}\right)$. Does it mean that the field at $(t_2, 0, 0, z_2(t_2))$ depends on the state of the first charge's motion at the advanced instant $t_1^{\rm adv}(t_2)$? It is not true because we study the interference of *outgoing* waves at time t (see figure 6). The moment $t_1^{\rm adv}(t_2)$ occurs before the observation instant t.

The third integral in between the first braces looks even more exotic than the 'advanced' one. Indeed, the hyperplane Σ_t seems to be a mirror for rays emanating from $(t_1', 0, 0, z_1(t_1'))$ (see figure 7). And yet the retarded causality is not violated. We still consider the interference of outgoing waves present at the observation time t.

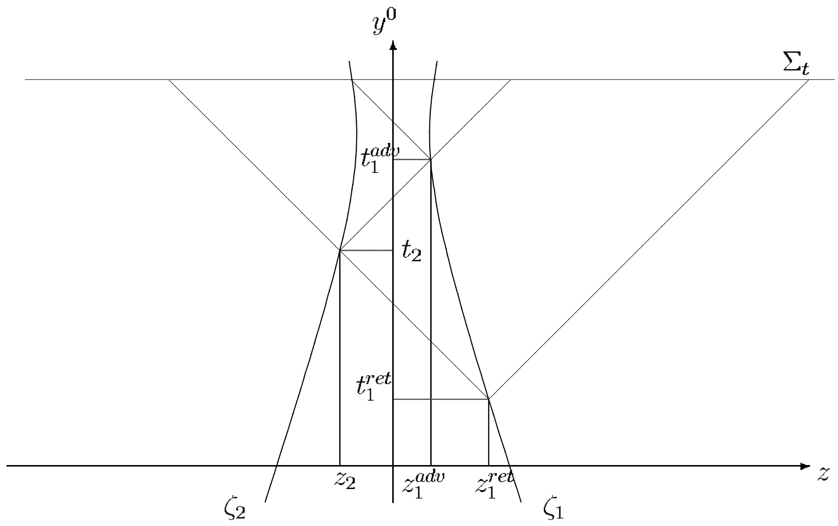

Figure 6. It seems that the field at $(t_2, 0, 0, z_2)$ depends on the state of the first charge motion at the retarded instant $t_1^{\rm ret}(t_2)$ as well as at the advanced instant $t_1^{\rm adv}(t_2)$. But we study the interference of retarded electromagnetic fields at the observation instant t. Both the moments $t_1^{\rm ret}(t_2)$ and $t_1^{\rm adv}(t_2)$ are before t.

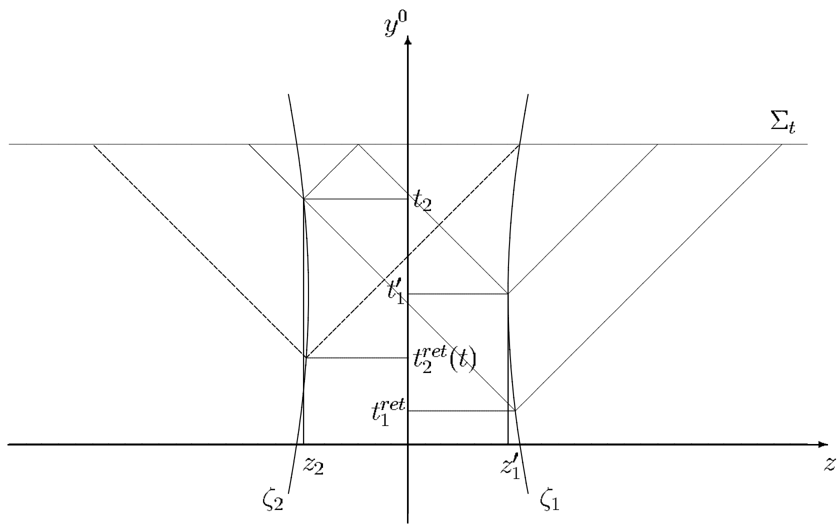

Figure 7. The ends of the interval $t_1 \in [t_1^{\rm ret}(t_2), t_1'(t, t_2)]$ are valuable in integration of the third integral in between the first braces in equation (5.7) (see integration rule (5.5)). The maximal value labels the vertex of the future light cone which touches a given forward light cone of $(t_2, 0, 0, z_2)$ at the observation instant t.

If one interchanges the words 'first particle' and 'second particle' in the above arguments we obtain the description of integrals written in between the second braces in equation (5.7). The situation is pictured in figures 2–5.

Our next task is to simplify the expression (5.7). Taking into account the third Wheeler and Feynman assumption ([6], p 160) we couple the half-retarded field of the second charge acting on the first source and its 'half-advanced response':

$$
\begin{aligned}
&\frac{e_2}{2}\int_{-\infty}^{t_2^{\rm ret}(t)} \mathrm{d}t_2\, F_{03}^{(1)}(\mathrm{adv}) + \frac{e_1}{2}\int_{-\infty}^{t} \mathrm{d}t_1 F_{03}^{(2)}(\mathrm{ret}) \\
&\qquad = -\frac{e_1e_2}{2}\left[\int_{-\infty}^{t_2^{\rm ret}(t)} \mathrm{d}t_2 \frac{1}{\left[z_1\left(t_1^{\rm adv}\right)-z_2(t_2)\right]^2}\frac{1+v_1\left(t_1^{\rm adv}\right)}{1-v_1\left(t_1^{\rm adv}\right)}\right. \\
&\qquad\quad \left.+\int_{-\infty}^{t} \mathrm{d}t_1 \frac{1}{\left[z_1(t_1)-z_2\left(t_2^{\rm ret}\right)\right]^2}\frac{1+v_2\left(t_2^{\rm ret}\right)}{1-v_2\left(t_2^{\rm ret}\right)}\right].
\end{aligned}
\tag{5.8}
$$

The crucial issue is that the functions $t_1^{\rm adv}(t_2)$ and $t_2^{\rm ret}(t_1)$ are inverses. It allows us to change the variables $\left(t_1^{\rm adv}(t_2), t_2\right) \mapsto \left(t_1, t_2^{\rm ret}(t_1)\right)$ in the 'advanced' integral. Equation (5.8) becomes

$$
\begin{aligned}
&-\frac{e_1e_2}{2}\left[\int_{-\infty}^{t} \mathrm{d}t_1 \frac{1}{\left[z_1(t_1)-z_2\left(t_2^{\rm ret}\right)\right]^2}\frac{1+v_1(t_1)}{1-v_2\left(t_2^{\rm ret}\right)} + \int_{-\infty}^{t} \mathrm{d}t_1 \frac{1}{\left[z_1(t_1)-z_2\left(t_2^{\rm ret}\right)\right]^2}\frac{1+v_2\left(t_2^{\rm ret}\right)}{1-v_2\left(t_2^{\rm ret}\right)}\right] \\
&\qquad = -\frac{e_1e_2}{2}\int_{-\infty}^{t} \mathrm{d}t_1 \frac{1}{\left[z_1(t_1)-z_2\left(t_2^{\rm ret}\right)\right]^2}\frac{v_1(t_1)-v_2\left(t_2^{\rm ret}\right)}{1-v_2\left(t_2^{\rm ret}\right)} \\
&\qquad\quad - e_1e_2\int_{-\infty}^{t} \mathrm{d}t_1 \frac{1}{\left[z_1(t_1)-z_2\left(t_2^{\rm ret}\right)\right]^2}\frac{1+v_2\left(t_2^{\rm ret}\right)}{1-v_2\left(t_2^{\rm ret}\right)} \\
&\qquad = \frac{1}{2}\left.\frac{e_1e_2}{z_1(t_1)-z_2\left(t_2^{\rm ret}\right)}\right|_{t_1\to-\infty}^{t_1=t} + e_1\int_{-\infty}^{t} \mathrm{d}t_1 F_{03}^{(2)}(\mathrm{ret}) \\
&\qquad = \frac{1}{2}\frac{e_1e_2}{z_1(t)-z_2\left[t_2^{\rm ret}(t)\right]} + e_1\int_{-\infty}^{t} \mathrm{d}t_1\, F_{03}^{(2)}(\mathrm{ret}).
\end{aligned}
\tag{5.9}
$$

Secondly, we join the half-retarded field of the first charge acting on the second source and its 'half-advanced response':

$$
\begin{aligned}
&\frac{e_2}{2}\int_{-\infty}^{t} \mathrm{d}t_2\, F_{03}^{(1)}(\mathrm{ret}) + \frac{e_1}{2}\int_{-\infty}^{t_1^{\rm ret}(t)} \mathrm{d}t_1 F_{03}^{(2)}(\mathrm{adv}) \\
&\qquad = \frac{e_1e_2}{2}\int_{-\infty}^{t} \mathrm{d}t_2 \frac{1}{\left[z_1\left(t_1^{\rm ret}\right)-z_2(t_2)\right]^2}\frac{1-v_1\left(t_1^{\rm ret}\right)}{1+v_1\left(t_1^{\rm ret}\right)} \\
&\qquad\quad + \frac{e_1e_2}{2}\int_{-\infty}^{t_1^{\rm ret}(t)} \mathrm{d}t_1 \frac{1}{\left[z_1(t_1)-z_2\left(t_2^{\rm adv}\right)\right]^2}\frac{1-v_2\left(t_2^{\rm adv}\right)}{1+v_2\left(t_2^{\rm adv}\right)} \\
&\qquad = -\frac{1}{2}\frac{e_1e_2}{z_1\left[t_1^{\rm ret}(t)\right]-z_2(t)} + e_2\int_{-\infty}^{t} \mathrm{d}t_2\, F_{03}^{(1)}(\mathrm{ret}).
\end{aligned}
\tag{5.10}
$$

The remaining terms constitute the integral being a function of the end points only:

$$
\begin{aligned}
&-\frac{e_1e_2}{2}\int_{t_2^{\rm ret}(t)}^{t} \mathrm{d}t_2 \frac{1}{[z_1(t_1')-z_2(t_2)]^2}\frac{1-v_1(t_1')}{1+v_1(t_1')} + \frac{e_1e_2}{2}\int_{t_1^{\rm ret}(t)}^{t} \mathrm{d}t_1 \frac{1}{[z_1(t_1)-z_2(t_2')]^2}\frac{1+v_2(t_2')}{1-v_2(t_2')} \\
&\qquad = \frac{e_1e_2}{2}\int_{t_1^{\rm ret}(t)}^{t} \mathrm{d}t_1 \frac{1}{[z_1(t_1)-z_2(t_2')]^2}\frac{v_1(t_1)+v_2(t_2')}{1-v_2(t_2')} \\
&\qquad = -\frac{1}{2}\frac{e_1e_2}{z_1(t)-z_2\left[t_2^{\rm ret}(t)\right]} + \frac{1}{2}\frac{e_1e_2}{z_1\left[t_1^{\rm ret}(t)\right]-z_2(t)}.
\end{aligned}
\tag{5.11}
$$

Summing up (5.9)–(5.11) we finally obtain the desired expression which does not contain advanced quantities:

$$p^3_{\rm int} = e_1 \int_{-\infty}^{t} {\rm d}t_1\, F^{(2)}_{03}({\rm ret}) + e_2 \int_{-\infty}^{t} {\rm d}t_2\, F^{(1)}_{03}({\rm ret}). \tag{5.12}$$

Therefore, the interference of outgoing electromagnetic waves leads to the interaction between the sources. The first charge seems a 'perfect absorber' for the radiation given off by the second one and vice versa.

5.1.1. Non-relativistic approximation. 'Self-action' contributions which arise due to volume integration of $T^{03}_{(1)}$ and $T^{03}_{(2)}$ are obtained in [4], equation (3.5). Apart from the relativistic Larmor terms, the third component of energy–momentum also contains the interference term (5.12):

$$p^3 = \sum_{a=1}^{2}\left[p^3_{a,{\rm part}} + \frac{2}{3}e_a^2 \int_{-\infty}^{t} {\rm d}t_a\, \boldsymbol{a}_a^2(t_a) v_a(t_a)\right] + e_1 \int_{-\infty}^{t} {\rm d}t_1\, F^{(2)}_{03}({\rm ret}) + e_2 \int_{-\infty}^{t} {\rm d}t_2\, F^{(1)}_{03}({\rm ret}). \tag{5.13}$$

In the non-relativistic approximation the momenta $p^3_{a,{\rm part}}$, $a = 1, 2$, become $m_a v_a(t)$, Larmor radiation terms are too small to be observed, and an action propagates instantaneously. It is easy to show that the interaction term (5.12) vanishes in this approximation (it accords with Newton's third law). The total momentum (5.13) of our two-particle system becomes the usual sum $m_1 v_1(t) + m_2 v_2(t)$ in the non-relativistic approximation.

5.2. Zeroth component

In this subsection we trace a series of stages in the calculation of the volume integral

$$p^0_{\rm int} = \int_{\Sigma_t} {\rm d}\sigma_0\, T^{00}_{\rm int}. \tag{5.14}$$

In appendix A we perform the computation in detail.

The interference part $T^{00}_{\rm int}$ of 'two-particle' energy density (2.5) follows by substituting equations (4.6) into equation (2.6). Taking into account the third line of (3.2), we obtain

$$4\pi T^{00}_{\rm int} = \frac{e_1 e_2}{(r_1 r_2)^3}\left\{a_1 a_2 \left[\frac{1}{2}(t-t_1)^2 + \frac{1}{2}(t-t_2)^2 - \frac{q^2}{2}\right] + r^2 b_1 b_2 - (t-t_1)(Z-z_2) b_1 a_2 \right.$$
$$\left. - (Z-z_1)(t-t_2) a_1 b_2 + (t-t_1)(t-t_2) b_1 b_2\right\}. \tag{5.15}$$

Routine scrupulous calculations allow us to express the integrand $\sqrt{-g}T^{00}_{\rm int}$ in the form of combinations of partial derivatives in retarded times (see appendix A). The double derivative

$$\frac{\partial^2}{\partial t_1 \partial t_2}\left(\frac{r^2}{q r_1 r_2} + \frac{k_1 k_2}{q}\right) \tag{5.16}$$

is involved. Here

$$k_1 = \frac{v_1(t-t_1) - (Z-z_1)}{r_1} \qquad k_2 = \frac{v_2(t-t_2) - (Z-z_2)}{r_2}. \tag{5.17}$$

The double derivative can be written in the form either

$$\frac{\partial}{\partial t_1}\left[\frac{\partial}{\partial t_2}\left(\frac{r^2}{q r_1 r_2} + \frac{k_1 k_2}{q}\right)\right] \tag{5.18}$$

or

$$\frac{\partial}{\partial t_2}\left[\frac{\partial}{\partial t_1}\left(\frac{r^2}{q r_1 r_2}+\frac{k_1 k_2}{q}\right)\right]. \tag{5.19}$$

There are two possible methods of integrating the expression (5.16). We can choose (5.19) and apply the integration rule (5.4) under the first charge mapping. The result is

$$p^0_{\rm int} = e_1\int_{-\infty}^{t} {\rm d}t_1 v_1(t_1) F^{(2)}_{03}({\rm ret}) + e_2\int_{-\infty}^{t} {\rm d}t_2 v_2(t_2) F^{(1)}_{03}({\rm ret}) - \frac{e_1 e_2}{z_1(t) - z_2\left[t_2^{\rm ret}(t)\right]}. \tag{5.20}$$

On the other hand, we can choose (5.18) and use the integration rule (5.5) under the second charge mapping. We obtain

$$p^0_{\rm int} = e_1\int_{-\infty}^{t} {\rm d}t_1\, v_1(t_1) F^{(2)}_{03}({\rm ret}) + e_2\int_{-\infty}^{t} {\rm d}t_2\, v_2(t_2) F^{(1)}_{03}({\rm ret}) - \frac{e_1 e_2}{z_1\left[t_1^{\rm ret}(t)\right] - z_2(t)}. \tag{5.21}$$

Otherwise, the calculations give the 'immovable core' which describes the action of the fields due to one charge on another (mutual interaction).

It is natural to interpret the 'changeable shell'

$$-\frac{e_1 e_2}{z_1(t) - z_2\left[t_2^{\rm ret}(t)\right]} \tag{5.22}$$

or

$$-\frac{e_1 e_2}{z_1\left[t_1^{\rm ret}(t)\right] - z_2(t)} \tag{5.23}$$

as the negative of interaction potential. The potential (5.22) is acausal: the first charge moves in the retarded field of the second one while the second particle moves in the advanced field of the first one. Similarly, the potential (5.23) ensures that the interaction can be both forward (2 to 1) and backward (1 to 2) in time. Such models were first elaborated by Staruszkiewicz [11]. The author shows that corresponding equations of motion can be reduced to ordinary differential equations. Recently [12–14], there has been considerable interest in the time-asymmetric model of the relativistic two-particle system.

5.2.1. Non-relativistic approximation. In the non-relativistic approximation a disturbance travels with infinite speed. The immovable part of $p^0_{\rm int}$ becomes the usual Coulomb potential:

$$\begin{aligned} p^0_{\rm int}\big|_{c\to\infty} &= -e_1 e_2\int_{-\infty}^{t} {\rm d}t\frac{v_1(t)}{[z_1(t) - z_2(t)]^2} + e_1 e_2\int_{-\infty}^{t} {\rm d}t\frac{v_2(t)}{[z_1(t) - z_2(t)]^2} \\ &= \frac{e_1 e_2}{z_1(t) - z_2(t)}. \end{aligned} \tag{5.24}$$

And the changeable 'shell' becomes the same potential taken with opposite signs! The sum is equal to zero; it is inconsistent with observation.

5.2.2. A very massive particle. Let the second charged particle be very massive, and the first one be light. We use the Lorentz frame where the massive particle is at rest, i.e. $v_2 = 0$. In this approximation the immovable part of $p^0_{\rm int}$ becomes the Coulomb potential too:

$$p^0_{\rm int} = -e_1 e_2\int_{-\infty}^{t} {\rm d}t_1\frac{v_1(t_1)}{[z_1(t_1) - z_{2,0}]^2} = \frac{e_1 e_2}{z_1(t) - z_{2,0}}. \tag{5.25}$$

The changeable terms, either

$$-\frac{e_1 e_2}{z_1(t) - z_{2,0}} \tag{5.26}$$

or

$$-\frac{e_1 e_2}{z_1\left[t_1^{\rm ret}(t)\right] - z_{2,0}} \tag{5.27}$$

distort the truth.

The integral of the bound part of the 'self-action' component $T^{0\nu}_{(a)}$ depends on the form of a spacelike three-surface over which the volume integration is performed [3]. The result is determined by the state of particle motion at the observation time only [3, 4]. While the radiative part of the energy–momentum carried by electromagnetic field is invariant, it is accumulated with time. Similarly, the immovable 'core' of $p^0_{\rm int}$ is a functional of particle paths while the changeable 'shell' consists of the functions of momentary positions of particles (delay in action ensures the shifts in arguments in expressions (5.22) and (5.23)). The following question now arises, if the 'shell' is a usual deformation of the bound electromagnetic 'cloud' which cannot be separated from a charged particle. The above approximations reinforce our conviction that the 'shell' expresses the deformation of electromagnetic 'clouds' of charged particles due to mutual interaction. Thus only the immovable terms should be taken into account.

6. Interference part of the total angular momentum of the electromagnetic field

We now turn to the calculation of the total angular momentum tensor of the electromagnetic field [5]:

$$M^{\mu\nu}_{\rm em} = \int_{\Sigma_t} {\rm d}\sigma_0\, (y^\mu T^{0\nu} - y^\nu T^{0\mu}). \tag{6.1}$$

Conservation of the space part $M^{ij}_{\rm em}$ of the tensor $M^{\mu\nu}_{\rm em}$ takes place due to invariance under space rotations. Conservation of the mixed spacetime components, $M^{0i}_{\rm em}$, is due to invariance under Lorentz transformations.

The structure of the electromagnetic field stress–energy tensor (2.5) suggests that the interference part of $M^{\mu\nu}_{\rm em}$ follows by substituting $T^{0\alpha}_{\rm int}$ for $T^{0\alpha}$ into (6.1).

6.1. Interference part of space components

Inserting the momentum densities (5.3) into

$$M^{ij}_{\rm int} = \int_{\Sigma_t} {\rm d}\sigma_0 \left(y^i T^{0j}_{\rm int} - y^j T^{0i}_{\rm int}\right) \tag{6.2}$$

and performing the coordinate transformation (3.3), we obtain the interference part of the angular momentum $M^{ij}_{\rm em}$. It is easy to show that integrand $y^1 T^{02}_{\rm int} - y^2 T^{01}_{\rm int}$ is equal to zero identically. The others, $y^1 T^{03}_{\rm int} - y^3 T^{01}_{\rm int}$ and $y^2 T^{03}_{\rm int} - y^3 T^{02}_{\rm int}$, are proportional to $\sin\varphi$ and $\cos\varphi$, respectively. They vanish due to angle integration. Therefore

$$J^k_{\rm int} := \varepsilon^k{}_{ij} M^{ij}_{\rm int} = 0. \tag{6.3}$$

6.2. Interference part of spacetime components

It is easy to check that the integrands $y^0 T^{01}_{\rm int} - y^1 T^{00}_{\rm int}$ and $y^0 T^{02}_{\rm int} - y^2 T^{00}_{\rm int}$ are proportional to $\sin\varphi$ and $\cos\varphi$, respectively. Thus, both the components $K^1_{\rm int} := -M^{01}_{\rm int}$ and $K^2_{\rm int} := -M^{02}_{\rm int}$ vanish due to angle integration. Only the third component

$$K^3_{\rm int} := -M^{03}_{\rm int} = \int_{\Sigma_t} {\rm d}\sigma_0 \left(-y^0 T^{03}_{\rm int} + y^3 T^{00}_{\rm int}\right) \tag{6.4}$$

is non-trivial.

The integration is performed in appendix B. Apart from the immovable 'core'

$$e_1 \int_{-\infty}^{t} \mathrm{d}t_1 \left(-t_1 F_{03}^{(2)}(\mathrm{ret}) + z_1(t_1) v_1(t_1) F_{03}^{(2)}(\mathrm{ret})\right) + e_2 \int_{-\infty}^{t} \mathrm{d}t_2 \left(-t_2 F_{03}^{(1)}(\mathrm{ret}) + z_2(t_2) v_2(t_2) F_{03}^{(1)}(\mathrm{ret})\right) \tag{6.5}$$

K_{int}^3 contains a changeable term, either

$$-z_1(t) \frac{e_1 e_2}{z_1(t) - z_2\left[t_2^{\mathrm{ret}}(t)\right]} \tag{6.6}$$

or

$$-z_2(t) \frac{e_1 e_2}{z_1\left[t_1^{\mathrm{ret}}(t)\right] - z_2(t)}. \tag{6.7}$$

For the reason substantiated in the previous section we omit the changeable term as unphysical.

7. Equations of motion and their interpretation

The structure of the Maxwell energy–momentum tensor density determines the structure of conserved quantities carried by electromagnetic field. The 'two-particle' electromagnetic field stress–energy tensor (2.5) consists of the 'individual' densities and the interference term (2.6). Thus a conserved quantity, say Ξ, contains, apart from (individual) 'self-action' terms, also a contribution from an interaction part:

$$\Xi = \Xi_{(1)} + \Xi_{(2)} + \Xi_{(12)}.$$

The 'self-action' expressions are obtained in [4] where a composite one-particle plus field system is considered.

The Lorentz force

$$F_{ab}^{\alpha} = e_b F_{(a)\beta}^{\alpha}(\mathrm{ret}) u_b^{\beta} \tag{7.1}$$

expresses the action of the (retarded) Liénard–Wiechert field due to charge a on charge b. u_b^{β} denotes the (normalized) four-velocity vector

$$\left(u_b^{\beta}\right) := \left(u_b^0, 0, 0, u_b^3\right) = \frac{1}{\sqrt{1 - v_b^2}}(1, 0, 0, v_b). \tag{7.2}$$

Substituting (7.1) into (5.13) yields

$$p^3 = \sum_{a=1}^{2} \left[p_{a,\mathrm{part}}^3 + \frac{2}{3} e_a^2 \int_{-\infty}^{t} \mathrm{d}t_a\, \boldsymbol{a}_a^2(t_a) v_a(t_a)\right] - \int_{-\infty}^{t} \mathrm{d}t_1 \sqrt{1 - v_1^2}\, F_{21}^3 - \int_{-\infty}^{t} \mathrm{d}t_2 \sqrt{1 - v_2^2}\, F_{12}^3. \tag{7.3}$$

Taking into account self-action expressions for energy ([4], equation (3.4)) and centre-of-mass conserved quantity ([4], equation (3.7)), we obtain

$$p^0 = \sum_{a=1}^{2} \left[p_{a,\mathrm{part}}^0 + \frac{2}{3} e_a^2 \int_{-\infty}^{t} \mathrm{d}t_a\, \boldsymbol{a}_a^2(t_a)\right] - \int_{-\infty}^{t} \mathrm{d}t_1 \sqrt{1 - v_1^2}\, F_{21}^0 - \int_{-\infty}^{t} \mathrm{d}t_2 \sqrt{1 - v_2^2}\, F_{12}^0 \tag{7.4}$$

$$K^3 = \sum_{a=1}^{2}\left[-tp^3_{a,\text{part}} + z_a(t)p^0_{a,\text{part}} + \frac{2}{3}e_a^2\int_{-\infty}^{t}\mathrm{d}t_a\,\boldsymbol{a}_a^2(t_a)[z_a(t_a) - v_a(t_a)t_a]\right.$$
$$\left. + \frac{4}{3}e_a^2\int_{-\infty}^{t}\mathrm{d}t_a\frac{v_a^2\dot v_a}{\left(1-v_a^2\right)^2}\right] + \int_{-\infty}^{t}\mathrm{d}t_1\sqrt{1-v_1^2}\left[t_1F_{21}^3 - z_1(t_1)F_{21}^0\right]$$
$$+ \int_{-\infty}^{t}\mathrm{d}t_2\sqrt{1-v_2^2}\left[t_2F_{12}^3 - z_2(t_2)F_{12}^0\right] \tag{7.5}$$

where $\boldsymbol{a}_a^2 = \dot v_a^2/\left(1-v_a^2\right)^3$ is the square $a_\mu a^\mu$ of four-acceleration. It is easy to rewrite these conserved quantities in a manifestly covariant fashion.

We can construct equations of motion as follows. We calculate the total flows of electromagnetic field energy, momentum and angular momentum which flow across the hyperplane Σ_t. We can do it at a time $t + \triangle t$. The change in these quantities should be balanced by a corresponding change in those of the particles. Since the action is not propagated instantaneously, the balance in a vicinity of the first charge as well as in a neighbourhood of the second charge should be achieved separately. The analysis of (7.4) and (7.3) gives the relativistic generalization of Newton's second law

$$\dot p^0_{a,\text{part}} = -\tfrac{2}{3}e_a^2\boldsymbol{a}_a^2(t) + \sqrt{1-v_a^2}F^0_{ba} \qquad \dot p^3_{a,\text{part}} = -\tfrac{2}{3}e_a^2\boldsymbol{a}_a^2(t)v_a(t) + \sqrt{1-v_a^2}F^3_{ba} \tag{7.6}$$

where loss of energy due to radiation is taken into account. From the differentiation of (7.5) we arrive at the following equality:

$$-t\sum_{a\neq b}\left[\dot p^3_{a,\text{part}} + \frac{2}{3}e_a^2\boldsymbol{a}_a^2(t)v_a(t) - \sqrt{1-v_a^2}F^3_{ba}\right]$$
$$+ \sum_{a\neq b}z_a(t)\left[\dot p^0_{a,\text{part}} + \frac{2}{3}e_a^2\boldsymbol{a}_a^2(t) - \sqrt{1-v_a^2}F^0_{ba}\right] - p^3_{1,\text{part}} + v_1(t)p^0_{1,\text{part}}$$
$$+ \frac{4}{3}e_1^2\frac{v_1^2\dot v_1}{\left(1-v_1^2\right)^2} - p^3_{2,\text{part}} + v_2(t)p^0_{2,\text{part}} + \frac{4}{3}e_2^2\frac{v_2^2\dot v_2}{\left(1-v_2^2\right)^2} = 0. \tag{7.7}$$

Taking into account the relativistic generalization of Newton's second law (7.6), we see that the bracketed expressions vanish. The remaining part does not contain the Lorentz forces at all. Non-bracketed terms are of two types: (i) those evaluated at the first source, (ii) those associated with the second charge. They vanish separately,

$$p^3_{a,\text{part}} - v_a(t)p^0_{a,\text{part}} = \frac{4}{3}e_a^2\frac{v_a^2\dot v_a}{\left(1-v_a^2\right)^2} \tag{7.8}$$

(cf [4], equation (3.11)).

A frontal collision of two charges is a very specific event. Any other two-particle motion possesses non-trivial space components of total angular momentum. When analysing them we can get the equations ([4], (3.10)). Together with equations of type (7.8) they constitute the system of six linear equations in variables p^μ_{part} which does not possess a solution whenever particle motion is accelerated. That is why I think that the 'self-action' problem is still unsolved. Otherwise, the Teitelboim expression (1.1) for the four-momentum of accelerated charge does not satisfy the equality (7.8) which arises from the invariance of the system under Lorentz transformation.

8. Conclusions

In the present paper, we determine the total four-momentum and total angular momentum of a closed system of two point-like charged particles and electromagnetic fields. For the sake of simplicity, we consider a frontal collision of asymptotically free particles. Calculations are performed via the integration of energy and momentum densities over three-dimensional hyperplane $y^0 = \text{const}$. The crucial issue is that the Maxwell energy–momentum tensor density is the sum of 'one-particle' densities and an 'interference' term (see equation (2.5)). Therefore, the conserved quantities consist of individual 'self-action' terms and interaction terms. Thus, the radiative part of the total four-momentum of a closed system of particles plus field contains, apart from the usual relativistic Larmor terms, also a contribution from the combination of the retarded Liénard–Wiechert fields. The latter is then nothing but the sum of work done by Lorentz forces of point-like charges acting on one another.

We can briefly summarize our conclusions as follows:

- an interference of outgoing electromagnetic waves (retarded Liénard–Wiechert fields) leads to the interaction between the sources;
- a point charge in otherwise charge-free space moves according to the law of inertia;
- a point charge within an interaction area radiates electromagnetic energy; the force of radiative reaction arises from the direct action of a particle upon itself.

The structure of conserved quantities of a closed system of particles plus field implies that the four-momentum of radiated charged particle is not proportional to its four-velocity. Teitelboim's expression for the particle four-momentum as a linear function of the four-velocity and four-acceleration is inconsistent with the derivative of the 'centre-of-mass' conserved quantity. Therefore, the 'renormalization of four-momentum' cannot be reduced to the commonly used 'renormalization of mass'.

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Appendix A. Volume integration of the interference part of energy density

There is a chain of identities obtained via differentiation of (3.1):

$$
\begin{aligned}
&\frac{\partial Z}{\partial t_1} = \frac{t - t_1 - (Z - z_1)v_1}{q} := \frac{r_1}{q} \\
&\frac{\partial Z}{\partial t_2} = -\frac{t - t_2 - (Z - z_2)v_2}{q} := -\frac{r_2}{q} \\
&\frac{\partial r_1}{\partial t_1} = -1 + v_1^2 - \frac{r_1}{q}v_1 - (Z - z_1)\dot{v}_1 \qquad \frac{\partial r_1}{\partial t_2} = \frac{r_2}{q}v_1 \\
&\frac{\partial r_2}{\partial t_1} = -\frac{r_1}{q}v_2 \qquad \frac{\partial r_2}{\partial t_2} = -1 + v_2^2 + \frac{r_2}{q}v_2 - (Z - z_2)\dot{v}_2.
\end{aligned}
\tag{A.1}
$$

Hence one has again

$$
\frac{a_b}{qr_b^2} = \frac{\partial}{\partial t_b}\left(\frac{1}{qr_b}\right) \qquad \frac{b_a}{qr_a^2} = \frac{\partial}{\partial t_a}\left(\frac{v_a}{qr_a}\right). \tag{A.2}
$$

Having used these identities we rewrite the expression $\sqrt{-g}T^{00}_{\rm int}$ in terms of the partial derivatives with respect to the retarded times t_1 and t_2. For example, we transform the first term in (5.15) as follows:

$$
\begin{aligned}
\frac{1}{2}(t-t_1)^2\frac{a_1a_2}{qr_1^2r_2^2} &= \frac{1}{2}(t-t_1)^2\frac{a_1}{r_1^2}\frac{\partial}{\partial t_2}\left(\frac{1}{qr_2}\right)\\
&= \frac{\partial}{\partial t_2}\left[\frac{1}{2}(t-t_1)^2\frac{a_1}{qr_1^2r_2}\right] - \frac{(t-t_1)^2}{2qr_2}\frac{\partial}{\partial t_2}\left(\frac{a_1}{r_1^2}\right)\\
&= \frac{\partial}{\partial t_2}\left[\frac{1}{2}(t-t_1)^2\frac{a_1}{qr_1^2r_2}\right] + \frac{1}{2}(t-t_1)^2\frac{\partial}{\partial t_1}\left(\frac{v_1}{q^2r_1^2}\right)\\
&= \frac{\partial}{\partial t_2}\left[\frac{1}{2}(t-t_1)^2\frac{a_1}{qr_1^2r_2}\right] + \frac{\partial}{\partial t_1}\left[\frac{1}{2}(t-t_1)^2\frac{v_1}{q^2r_1^2}\right] + (t-t_1)\frac{v_1}{q^2r_1^2}. \qquad (A.3)
\end{aligned}
$$

The term under $\partial/\partial t_2$ takes the form

$$
\frac{1}{2}(t-t_1)^2\frac{a_1}{qr_1^2r_2} = \frac{\partial}{\partial t_1}\left[\frac{(t-t_1)^2}{2qr_1r_2}\right] + \frac{t-t_1}{qr_1r_2} - \frac{1}{2}(t-t_1)^2\frac{v_2}{q^2r_2^2}.
$$

On rearrangement, the final transformation looks as follows:

$$
\begin{aligned}
\frac{1}{2}(t-t_1)^2\frac{a_1a_2}{qr_1^2r_2^2} &= \frac{\partial^2}{\partial t_2\partial t_1}\left[\frac{(t-t_1)^2}{2qr_1r_2}\right] + \frac{\partial}{\partial t_2}\left[\frac{t-t_1}{qr_1r_2} - \frac{1}{2}(t-t_1)^2\frac{v_2}{q^2r_2^2}\right]\\
&\quad + \frac{\partial}{\partial t_1}\left[\frac{1}{2}(t-t_1)^2\frac{v_1}{q^2r_1^2}\right] + (t-t_1)\frac{v_1}{q^2r_1^2}. \qquad (A.4)
\end{aligned}
$$

As another example, we consider the term

$$
\begin{aligned}
\frac{r^2b_1b_2}{qr_1^2r_2^2} &= \frac{\partial}{\partial t_1}\left[r^2\frac{v_1b_2}{qr_1r_2^2}\right] + \frac{\partial}{\partial t_2}\left[v_1\frac{(t-t_2)^2\left[1-v_2^2\right]}{q^2r_2^2}\right]\\
&= \frac{\partial}{\partial t_2}\left[r^2\frac{v_2b_1}{qr_1^2r_2}\right] - \frac{\partial}{\partial t_1}\left[v_2\frac{(t-t_1)^2\left[1-v_1^2\right]}{q^2r_1^2}\right] \qquad (A.5)
\end{aligned}
$$

which does not contain a 'remnant' in addition to partial derivatives (cf (A.4)).

The remaining terms constitute a polynomial of analogous structure:

$$
\begin{aligned}
&\frac{1}{qr_1^2r_2^2}\left\{a_1a_2\left[\frac{1}{2}(t-t_1)^2 + \frac{1}{2}(t-t_2)^2 - \frac{q^2}{2}\right] - (t-t_1)(Z-z_2)b_1a_2\right.\\
&\qquad\left. - (Z-z_1)(t-t_2)a_1b_2 + (t-t_1)(t-t_2)b_1b_2\right\}\\
&\quad = \frac{\partial}{\partial t_1}\left[r^2\frac{v_1}{q^2r_1^2} + \frac{v_1r_2}{q^2r_1}k_1k_2 + \frac{1}{qr_1} - \frac{v_1(t-t_1)+v_2(t-t_2)}{q^2r_1}\right]\\
&\qquad + \frac{\partial}{\partial t_2}\left[-r^2\frac{v_2}{q^2r_2^2} - \frac{v_2r_1}{q^2r_2}k_1k_2 + \frac{1}{qr_2} + \frac{v_1(t-t_1)+v_2(t-t_2)}{q^2r_2}\right]\\
&\qquad + \frac{\partial^2}{\partial t_1\partial t_2}\left[\frac{r^2}{qr_1r_2} + \frac{k_1k_2}{q}\right]\\
&\quad := \frac{\partial G_1}{\partial t_1} + \frac{\partial G_2}{\partial t_2} + \frac{\partial^2 G}{\partial t_1\partial t_2} \qquad (A.6)
\end{aligned}
$$

where k_1 and k_2 are defined by equations (5.17). (Prefactor $e_1e_2/4\pi$ is omitted.)

Now we turn to volume integration. It is defined in subsection 5.1 of the present paper.

The terms which are proportional to r^2 are not essential in integration procedure. The reason is that the radius (3.2) is equal to zero at the end points (3.4)–(3.11) of integrals. So, the valuable part of (A.5) can be expressed as either

$$e_1e_2\frac{\partial}{\partial t_2}\left[v_1\frac{(t-t_2)^2\left[1-v_2^2\right]}{q^2r_2^2}\right] \tag{A.7}$$

or

$$-e_1e_2\frac{\partial}{\partial t_1}\left[v_2\frac{(t-t_1)^2\left[1-v_1^2\right]}{q^2r_1^2}\right]. \tag{A.8}$$

To calculate the integral of the partial derivative in t_2 we use the integration rule (5.4):

$$\begin{aligned}
&\left[\int_{-\infty}^{t_1^{\rm ret}(t)}\mathrm{d}t_1\int_{t_2^{\rm ret}(t_1)}^{t_2^{\rm adv}(t_1)}\mathrm{d}t_2+\int_{t_1^{\rm ret}(t)}^{t}\mathrm{d}t_1\int_{t_2^{\rm ret}(t_1)}^{t_2'(t,t_1)}\mathrm{d}t_2\right]\int_0^{2\pi}\mathrm{d}\varphi\, e_1e_2\frac{\partial}{\partial t_2}\left[v_1\frac{(t-t_2)^2\left[1-v_2^2\right]}{q^2r_2^2}\right]\\
&\qquad=2\pi e_1e_2\left[\int_{-\infty}^{t_1^{\rm ret}(t)}\mathrm{d}t_1\frac{v_1(t_1)}{q^2\big(t_1,t_2^{\rm adv}\big)}\frac{1-v_2\big(t_2^{\rm adv}\big)}{1+v_2\big(t_2^{\rm adv}\big)}\right.\\
&\qquad\qquad\left.-\int_{-\infty}^{t}\mathrm{d}t_1\frac{v_1(t_1)}{q^2\big(t_1,t_2^{\rm ret}\big)}\frac{1+v_2\big(t_2^{\rm ret}\big)}{1-v_2\big(t_2^{\rm ret}\big)}+\int_{t_1^{\rm ret}(t)}^{t}\mathrm{d}t_1\frac{v_1(t_1)}{q^2(t_1,t_2')}\frac{1+v_2(t_2')}{1-v_2(t_2')}\right].
\end{aligned}$$

To proceed further, it is suitable to perform the changes of variables $\big(t_1,t_2^{\rm adv}(t_1)\big)\mapsto\big(t_1^{\rm ret}(t_2),t_2\big)$ in the first integral and $(t_1,t_2'(t,t_1))\mapsto(t_1'(t,t_2),t_2)$ in the third integral. We arrive at

$$\begin{aligned}
&2\pi e_1e_2\left[\int_{-\infty}^{t}\mathrm{d}t_2\frac{v_1\big(t_1^{\rm ret}\big)}{q^2\big(t_1^{\rm ret},t_2\big)}\frac{1-v_2(t_2)}{1+v_1\big(t_1^{\rm ret}\big)}-\int_{-\infty}^{t}\mathrm{d}t_1\frac{v_1(t_1)}{q^2\big(t_1,t_2^{\rm ret}\big)}\frac{1+v_2\big(t_2^{\rm ret}\big)}{1-v_2\big(t_2^{\rm ret}\big)}\right.\\
&\qquad\left.+\int_{t_2^{\rm ret}(t)}^{t}\mathrm{d}t_2\frac{v_1(t_1')}{q^2(t_1',t_2)}\frac{1+v_2(t_2)}{1+v_1(t_1')}\right].
\end{aligned} \tag{A.9}$$

We use the rule (5.5) to find the integral of (A.8):

$$\begin{aligned}
&\left[\int_{-\infty}^{t_2^{\rm ret}(t)}\mathrm{d}t_2\int_{t_1^{\rm ret}(t_2)}^{t_1^{\rm adv}(t_2)}\mathrm{d}t_1+\int_{t_2^{\rm ret}(t)}^{t}\mathrm{d}t_2\int_{t_1^{\rm ret}(t_2)}^{t_1'(t,t_2)}\mathrm{d}t_1\right]\int_0^{2\pi}\mathrm{d}\varphi\, e_1e_2\frac{\partial}{\partial t_1}\left[-v_2\frac{(t-t_1)^2\left[1-v_1^2\right]}{q^2r_1^2}\right]\\
&\qquad=2\pi e_1e_2\left[-\int_{-\infty}^{t_2^{\rm ret}(t)}\mathrm{d}t_2\frac{v_2(t_2)}{q^2\big(t_1^{\rm adv},t_2\big)}\frac{1+v_1\big(t_1^{\rm adv}\big)}{1-v_1\big(t_1^{\rm adv}\big)}\right.\\
&\qquad\qquad\left.+\int_{-\infty}^{t}\mathrm{d}t_2\frac{v_2(t_2)}{q^2\big(t_1^{\rm ret},t_2\big)}\frac{1-v_1\big(t_1^{\rm ret}\big)}{1+v_1\big(t_1^{\rm ret}\big)}-\int_{t_2^{\rm ret}(t)}^{t}\mathrm{d}t_2\frac{v_2(t_2)}{q^2(t_1',t_2)}\frac{1-v_1(t_1')}{1+v_1(t_1')}\right].
\end{aligned}$$

The change of variables $\big(t_1^{\rm adv}(t_2),t_2\big)\mapsto\big(t_1,t_2^{\rm ret}(t_1)\big)$ in the first integral is necessary. Thus we obtain

$$\begin{aligned}
&2\pi e_1e_2\left[-\int_{-\infty}^{t}\mathrm{d}t_1\frac{v_2\big(t_2^{\rm ret}\big)}{q^2\big(t_1,t_2^{\rm ret}\big)}\frac{1+v_1(t_1)}{1-v_2\big(t_2^{\rm ret}\big)}+\int_{-\infty}^{t}\mathrm{d}t_2\frac{v_2(t_2)}{q^2\big(t_1^{\rm ret},t_2\big)}\frac{1-v_1\big(t_1^{\rm ret}\big)}{1+v_1\big(t_1^{\rm ret}\big)}\right.\\
&\qquad\left.-\int_{t_2^{\rm ret}(t)}^{t}\mathrm{d}t_2\frac{v_2(t_2)}{q^2(t_1',t_2)}\frac{1-v_1(t_1')}{1+v_1(t_1')}\right].
\end{aligned} \tag{A.10}$$

Subtracting (A.9) from (A.10), we see that these expressions are equal to each other:

$$\int_{-\infty}^{t} \mathrm{d}t_1 \frac{1}{q^2\left(t_1, t_2^{\mathrm{ret}}\right)} \frac{v_1(t_1) - v_2\left(t_2^{\mathrm{ret}}\right)}{1 - v_2\left(t_2^{\mathrm{ret}}\right)} + \int_{-\infty}^{t} \mathrm{d}t_2 \frac{1}{q^2\left(t_1^{\mathrm{ret}}, t_2\right)} \frac{-v_1\left(t_1^{\mathrm{ret}}\right) + v_2(t_2)}{1 + v_1\left(t_1^{\mathrm{ret}}\right)}$$
$$+ \int_{t_2^{\mathrm{ret}}(t)}^{t} \mathrm{d}t_2 \frac{1}{q^2(t_1', t_2)} \frac{-v_1(t_1') - v_2(t_2)}{1 + v_1(t_1')} = - \left. \frac{1}{z_1(t_1) - z_2\left(t_2^{\mathrm{ret}}\right)} \right|_{t_1 \to -\infty}^{t_1 = t}$$
$$+ \left. \frac{1}{z_1\left(t_1^{\mathrm{ret}}\right) - z_2(t_2)} \right|_{t_2 \to -\infty}^{t_2 = t} - \left. \frac{1}{z_1(t_1') - z_2(t_2)} \right|_{t_2 = t_2^{\mathrm{ret}}(t)}^{t_2 = t} = 0.$$

(Prefactor $2\pi e_1 e_2$ is omitted.)

The only way to deal with the sum of first-order partial derivatives involved in (A.6) is the method presented in subsection 5.1 of the present paper. Scrupulous calculations give

$$\int_{\Sigma_t} \mathrm{d}\sigma_0 \left(\frac{\partial G_1}{\partial t_1} + \frac{\partial G_2}{\partial t_2} \right)$$
$$= 2\pi e_1 e_2 \left[\int_{-\infty}^{t} \mathrm{d}t_1 \mathcal{A}\left(t_1, t_2^{\mathrm{ret}}\right) + \int_{-\infty}^{t} \mathrm{d}t_2 \mathcal{B}\left(t_1^{\mathrm{ret}}, t_2\right) + \int_{t_2^{\mathrm{ret}}(t)}^{t} \mathrm{d}t_2 \mathcal{C}(t_1', t_2) \right] \tag{A.11}$$

where

$$\begin{aligned}
\mathcal{A}\left(t_1, t_2^{\mathrm{ret}}\right) &= -\frac{(1 - v_1(t_1))\left(1 + v_2\left(t_2^{\mathrm{ret}}\right)\right)}{q\left(t_1, t_2^{\mathrm{ret}}\right)\left(t - t_2^{\mathrm{ret}}(t_1)\right)\left(1 - v_2\left(t_2^{\mathrm{ret}}\right)\right)} + \frac{1 + v_1(t_1)}{q\left(t_1, t_2^{\mathrm{ret}}\right)(t - t_1)} \\
&\quad + \frac{1}{q^2\left(t_1, t_2^{\mathrm{ret}}\right)} \frac{v_1(t_1) - v_2\left(t_2^{\mathrm{ret}}\right)}{1 - v_2\left(t_2^{\mathrm{ret}}\right)} - 2v_1(t_1) \frac{1}{q^2\left(t_1, t_2^{\mathrm{ret}}\right)} \frac{1 + v_2\left(t_2^{\mathrm{ret}}\right)}{1 - v_2\left(t_2^{\mathrm{ret}}\right)} \\
\mathcal{B}\left(t_1^{\mathrm{ret}}, t_2\right) &= -\frac{\left(1 - v_1\left(t_1^{\mathrm{ret}}\right)\right)(1 + v_2(t_2))}{q\left(t_1^{\mathrm{ret}}, t_2\right)\left(t - t_1^{\mathrm{ret}}(t_2)\right)\left(1 + v_1\left(t_1^{\mathrm{ret}}\right)\right)} + \frac{1 - v_2(t_2)}{q\left(t_1^{\mathrm{ret}}, t_2\right)(t - t_2)} \\
&\quad + \frac{1}{q^2\left(t_1^{\mathrm{ret}}, t_2\right)} \frac{v_1\left(t_1^{\mathrm{ret}}\right) - v_2(t_2)}{1 + v_1\left(t_1^{\mathrm{ret}}\right)} + 2v_2(t_2) \frac{1}{q^2\left(t_1^{\mathrm{ret}}, t_2\right)} \frac{1 - v_1\left(t_1^{\mathrm{ret}}\right)}{1 + v_1\left(t_1^{\mathrm{ret}}\right)} \\
\mathcal{C}(t_1', t_2) &= \frac{(1 - v_1(t_1'))(1 - v_2(t_2))}{q(t_1', t_2)(t - t_1'(t, t_2))(1 + v_1(t_1'))} + \frac{1 + v_2(t_2)}{q(t_1', t_2)(t - t_2)} \\
&\quad + \frac{1}{q^2(t_1', t_2)} \frac{-v_1(t_1') + v_2(t_2) - 2v_1(t_1')v_2(t_2)}{1 + v_1(t_1')}.
\end{aligned} \tag{A.12}$$

The result involves divergent terms (those proportional to $1/(t - t_1)$, $1/(t - t_2)$ and $1/(t - t_1'))$.

Let us consider the double derivative term (see equation (A.6)). Its contribution can be computed in two different ways. For the first term of the expression under the double derivative we have

$$\int_{\Sigma_t} \mathrm{d}\sigma_0 \frac{\partial^2}{\partial t_1 \partial t_2} \left(e_1 e_2 \frac{r^2}{q r_1 r_2} \right) = \left[\int_{-\infty}^{t_1^{\mathrm{ret}}(t)} \mathrm{d}t_1 \int_{t_2^{\mathrm{ret}}(t_1)}^{t_2^{\mathrm{adv}}(t_1)} \mathrm{d}t_2 + \int_{t_1^{\mathrm{ret}}(t)}^{t} \mathrm{d}t_1 \int_{t_2^{\mathrm{ret}}(t_1)}^{t_2'(t,t_1)} \mathrm{d}t_2 \right]$$
$$\times \int_0^{2\pi} \mathrm{d}\varphi \, e_1 e_2 \frac{\partial}{\partial t_2} \left[-\frac{2(Z - z_2)}{q^2 r_2} \right]$$

$$= 2\pi e_1 e_2 \left[\int_{-\infty}^{t} \mathrm{d}t_1 \frac{1}{q^2\left(t_1, t_2^{\mathrm{ret}}\right)} \frac{2}{1 - v_2\left(t_2^{\mathrm{ret}}\right)} + \int_{-\infty}^{t} \mathrm{d}t_2 \frac{1}{q^2\left(t_1^{\mathrm{ret}}, t_2\right)} \frac{2}{1 + v_1\left(t_1^{\mathrm{ret}}\right)} \right.$$
$$\left. - \int_{t_2^{\mathrm{ret}}(t)}^{t} \mathrm{d}t_2 \frac{1}{q^2(t_1', t_2)} \frac{2}{1 + v_1(t_1')} \right]$$
$$= \left[\int_{-\infty}^{t_2^{\mathrm{ret}}(t)} \mathrm{d}t_2 \int_{t_1^{\mathrm{ret}}(t_2)}^{t_1^{\mathrm{adv}}(t_2)} \mathrm{d}t_1 + \int_{t_2^{\mathrm{ret}}(t)}^{t} \mathrm{d}t_2 \int_{t_1^{\mathrm{ret}}(t_2)}^{t_1'(t,t_2)} \mathrm{d}t_1 \right] \int_0^{2\pi} \mathrm{d}\varphi\, e_1 e_2 \frac{\partial}{\partial t_1} \left[\frac{2(Z - z_1)}{q^2 r_1} \right]. \tag{A.13}$$

Both the integration rules under the first charge mapping and the second charge mapping lead to the same result, while the contribution of the remaining part

$$\frac{\partial^2}{\partial t_1 \partial t_2} \left(\frac{k_1 k_2}{q} \right) = \frac{\partial}{\partial t_2} \left[\frac{1}{q} \frac{\partial}{\partial t_1} k_1 k_2 + k_1 k_2 \frac{\partial}{\partial t_1} \frac{1}{q} \right]$$
$$= \frac{\partial}{\partial t_1} \left[\frac{1}{q} \frac{\partial}{\partial t_2} k_1 k_2 + k_1 k_2 \frac{\partial}{\partial t_2} \frac{1}{q} \right] \tag{A.14}$$

substantially depends on the choice of integration rule (5.4) or (5.5).

The integration of the first term in between the square brackets gives the immovable 'core' of this contribution:

$$\left[\int_{-\infty}^{t_1^{\mathrm{ret}}(t)} \mathrm{d}t_1 \int_{t_2^{\mathrm{ret}}(t_1)}^{t_2^{\mathrm{adv}}(t_1)} \mathrm{d}t_2 + \int_{t_1^{\mathrm{ret}}(t)}^{t} \mathrm{d}t_1 \int_{t_2^{\mathrm{ret}}(t_1)}^{t_2'(t,t_1)} \mathrm{d}t_2 \right] \int_0^{2\pi} \mathrm{d}\varphi\, e_1 e_2 \frac{\partial}{\partial t_2} \left[\frac{1}{q} \frac{\partial}{\partial t_1} k_1 k_2 \right]$$
$$= 2\pi e_1 e_2 \left[\int_{-\infty}^{t} \mathrm{d}t_1 \mathcal{D}_0\left(t_1, t_2^{\mathrm{ret}}\right) + \int_{-\infty}^{t} \mathrm{d}t_2 \mathcal{E}_0\left(t_1^{\mathrm{ret}}, t_2\right) + \int_{t_2^{\mathrm{ret}}(t)}^{t} \mathrm{d}t_2 \mathcal{F}_0(t_1', t_2) \right]$$
$$= \left[\int_{-\infty}^{t_2^{\mathrm{ret}}(t)} \mathrm{d}t_2 \int_{t_1^{\mathrm{ret}}(t_2)}^{t_1^{\mathrm{adv}}(t_2)} \mathrm{d}t_1 + \int_{t_2^{\mathrm{ret}}(t)}^{t} \mathrm{d}t_2 \int_{t_1^{\mathrm{ret}}(t_2)}^{t_1'(t,t_2)} \mathrm{d}t_1 \right]$$
$$\times \int_0^{2\pi} \mathrm{d}\varphi\, e_1 e_2 \frac{\partial}{\partial t_1} \left[\frac{1}{q} \frac{\partial}{\partial t_2} k_1 k_2 \right]. \tag{A.15}$$

The integrands

$$\mathcal{D}_0\left(t_1, t_2^{\mathrm{ret}}\right) = \frac{(1 - v_1(t_1))\left(1 + v_2\left(t_2^{\mathrm{ret}}\right)\right)}{q\left(t_1, t_2^{\mathrm{ret}}\right)\left(t - t_2^{\mathrm{ret}}(t_1)\right)\left(1 - v_2\left(t_2^{\mathrm{ret}}\right)\right)} - \frac{1 + v_1(t_1)}{q\left(t_1, t_2^{\mathrm{ret}}\right)(t - t_1)}$$
$$+ \frac{1}{q^2\left(t_1, t_2^{\mathrm{ret}}\right)} \frac{-2 + 2v_1(t_1) v_2\left(t_2^{\mathrm{ret}}\right)}{1 - v_2\left(t_2^{\mathrm{ret}}\right)}$$
$$\mathcal{E}_0\left(t_1^{\mathrm{ret}}, t_2\right) = \frac{\left(1 - v_1\left(t_1^{\mathrm{ret}}\right)\right)(1 + v_2(t_2))}{q\left(t_1^{\mathrm{ret}}, t_2\right)\left(t - t_1^{\mathrm{ret}}(t_2)\right)\left(1 + v_1\left(t_1^{\mathrm{ret}}\right)\right)} - \frac{1 - v_2(t_2)}{q\left(t_1^{\mathrm{ret}}, t_2\right)(t - t_2)}$$
$$+ \frac{1}{q^2\left(t_1^{\mathrm{ret}}, t_2\right)} \frac{-2 + 2v_1\left(t_1^{\mathrm{ret}}\right) v_2(t_2)}{1 + v_1\left(t_1^{\mathrm{ret}}\right)} \tag{A.16}$$
$$\mathcal{F}_0(t_1', t_2) = -\frac{(1 - v_1(t_1'))(1 - v_2(t_2))}{q(t_1', t_2)(t - t_1'(t, t_2))(1 + v_1(t_1'))} - \frac{1 + v_2(t_2)}{q(t_1', t_2)(t - t_2)}$$
$$+ \frac{1}{q^2(t_1', t_2)} \frac{2 + 2v_1(t_1') v_2(t_2)}{1 + v_1(t_1')}$$

annul the divergent terms in expressions (A.12).

The changeable 'shell' follows from the volume integration of the second term in (A.14). The calculation based on the integration rule (5.4) gives

$$\left[\int_{-\infty}^{t_1^{\rm ret}(t)} {\rm d}t_1 \int_{t_2^{\rm ret}(t_1)}^{t_2^{\rm adv}(t_1)} {\rm d}t_2 + \int_{t_1^{\rm ret}(t)}^{t} {\rm d}t_1 \int_{t_2^{\rm ret}(t_1)}^{t_2'(t,t_1)} {\rm d}t_2\right] \int_0^{2\pi} {\rm d}\varphi\, e_1 e_2 \frac{\partial}{\partial t_2}\left[k_1 k_2 \frac{\partial}{\partial t_1}\frac{1}{q}\right]$$
$$= 2\pi e_1 e_2 \left[\int_{-\infty}^{t} {\rm d}t_1 \frac{v_1(t_1)}{q^2\left(t_1, t_2^{\rm ret}\right)} - \int_{-\infty}^{t} {\rm d}t_2 \frac{v_1\left(t_1^{\rm ret}\right)}{q^2\left(t_1^{\rm ret}, t_2\right)} \frac{1+v_2(t_2)}{1+v_1\left(t_1^{\rm ret}\right)}\right.$$
$$\left. + \int_{t_2^{\rm ret}(t)}^{t} {\rm d}t_2 \frac{v_1(t_1')}{q^2(t_1', t_2)} \frac{1-v_2(t_2)}{1+v_1(t_1')}\right]$$
$$= 2\pi e_1 e_2 \left[\int_{-\infty}^{t} {\rm d}t_1 \mathcal{D}_1\left(t_1, t_2^{\rm ret}\right) + \int_{-\infty}^{t} {\rm d}t_2 \mathcal{E}_1\left(t_1^{\rm ret}, t_2\right) + \int_{t_2^{\rm ret}(t)}^{t} {\rm d}t_2 \mathcal{F}_1(t_1', t_2)\right]. \tag{A.17}$$

The integration by means of (5.5) leads to

$$\left[\int_{-\infty}^{t_2^{\rm ret}(t)} {\rm d}t_2 \int_{t_1^{\rm ret}(t_2)}^{t_1^{\rm adv}(t_2)} {\rm d}t_1 + \int_{t_2^{\rm ret}(t)}^{t} {\rm d}t_2 \int_{t_1^{\rm ret}(t_2)}^{t_1'(t,t_2)} {\rm d}t_1\right] \int_0^{2\pi} {\rm d}\varphi\, e_1 e_2 \frac{\partial}{\partial t_1}\left[k_1 k_2 \frac{\partial}{\partial t_2}\frac{1}{q}\right]$$
$$= 2\pi e_1 e_2 \left[\int_{-\infty}^{t} {\rm d}t_1 \frac{v_2(t_2)}{q^2\left(t_1, t_2^{\rm ret}\right)} \frac{1-v_1(t_1)}{1-v_2\left(t_2^{\rm ret}\right)}\right.$$
$$\left. - \int_{-\infty}^{t} {\rm d}t_2 \frac{v_2(t_2)}{q^2\left(t_1^{\rm ret}, t_2\right)} - \int_{t_2^{\rm ret}(t)}^{t} {\rm d}t_2 \frac{v_2(t_2)}{q^2(t_1', t_2)}\right]$$
$$= 2\pi e_1 e_2 \left[\int_{-\infty}^{t} {\rm d}t_1\, \mathcal{D}_2\left(t_1, t_2^{\rm ret}\right) + \int_{-\infty}^{t} {\rm d}t_2\, \mathcal{E}_2\left(t_1^{\rm ret}, t_2\right) + \int_{t_2^{\rm ret}(t)}^{t} {\rm d}t_2\, \mathcal{F}_2(t_1', t_2)\right]. \tag{A.18}$$

Subtracting (A.17) from (A.18), we obtain the paradoxical equality

$$\frac{e_1 e_2}{z_1(t) - z_2\left[t_2^{\rm ret}(t)\right]} = \frac{e_1 e_2}{z_1\left[t_1^{\rm ret}(t)\right] - z_2(t)}.$$

(Factor 2π is omitted.) Therefore, expression (A.17) is not equal to expression (A.18).

Summing up all the contributions (A.11), (A.13), (A.15), (A.17), and either (A.10) or (A.9), we obtain the expression (5.20) for the interference part of energy. Substituting (A.18) for (A.17), we arrive at the expression (5.21).

Appendix B. Volume integration of the interference part of the third component of the 'centre-of-mass' density

Having done the coordinate transformation (3.3) in (6.4), we obtain

$$K_{\rm int}^3 = -t\int_{\Sigma_t} {\rm d}\sigma_0\, T_{\rm int}^{03} + \int_{\Sigma_t} {\rm d}\sigma_0\, Z T_{\rm int}^{00}. \tag{B.1}$$

Volume integration of the interference part of the third component of the electromagnetic field momentum density is performed in subsection 5.1 of the present paper (see final expression

(5.12)). To compute the integral of $ZT^{00}_{\rm int}$ we rewrite the integrand

$$\sqrt{-g}ZT^{00}_{\rm int} = \frac{e_1e_2}{4\pi}\frac{Z}{qr_1^2r_2^2}\left\{r^2b_1b_2 + a_1a_2\left[\frac{1}{2}(t-t_1)^2 + \frac{1}{2}(t-t_2)^2 - \frac{q^2}{2}\right]\right.$$
$$\left. -(t-t_1)(Z-z_2)b_1a_2 - (Z-z_1)(t-t_2)a_1b_2 + (t-t_1)(t-t_2)b_1b_2\right\} \qquad \text{(B.2)}$$

as a sum of partial derivatives in retarded times t_1 and t_2. The first term can be rewritten in the form

$$\frac{Z}{qr_1^2r_2^2}r^2b_1b_2 = \frac{\partial}{\partial t_1}\left[-Zv_2\frac{(t-t_1)^2\left(1-v_1^2\right)}{q^2r_1^2} + r^2\frac{v_1v_2}{q^2r_1} - \frac{v_2(Z-z_2)}{q^2}\right]$$
$$+\frac{\partial}{\partial t_2}\left[Zr^2\frac{v_2b_1}{qr_1^2r_2} + \frac{t-t_1}{q^2}\right] \qquad \text{(B.3)}$$

or

$$\frac{Z}{qr_1^2r_2^2}r^2b_1b_2 = \frac{\partial}{\partial t_2}\left[Zv_1\frac{(t-t_2)^2\left(1-v_2^2\right)}{q^2r_2^2} - r^2\frac{v_1v_2}{q^2r_2} + \frac{v_1(Z-z_1)}{q^2}\right]$$
$$+\frac{\partial}{\partial t_1}\left[Zr^2\frac{v_1b_2}{qr_1r_2^2} - \frac{t-t_2}{q^2}\right]. \qquad \text{(B.4)}$$

(Prefactor $e_1e_2/4\pi$ is omitted.) The volume integration can be handled via the rules (5.4) and (5.5). Taking the integrand in the form (B.3) we obtain

$$\int_{\Sigma_t}\mathrm{d}\sigma_0\, Z\frac{e_1e_2}{(r_1r_2)^3}r^2b_1b_1 = \frac{e_1e_2}{2}\left\{-\int_{-\infty}^{t}\mathrm{d}t_1\left[z_1(t_1)v_2\left(t_2^{\rm ret}\right)\frac{1+v_1(t_1)}{q^2\left(t_1,t_2^{\rm ret}\right)\left(1-v_2\left(t_2^{\rm ret}\right)\right)}\right.\right.$$
$$\left. +(t-t_1)\frac{1+v_2\left(t_2^{\rm ret}\right)}{q^2\left(t_1,t_2^{\rm ret}\right)\left(1-v_2\left(t_2^{\rm ret}\right)\right)} + \frac{v_2\left(t_2^{\rm ret}\right)(1-v_1(t_1))}{q\left(t_1,t_2^{\rm ret}\right)\left(1-v_2\left(t_2^{\rm ret}\right)\right)}\right]$$
$$+\int_{-\infty}^{t}\mathrm{d}t_2\left[z_2(t_2)v_2(t_2)\frac{1-v_1\left(t_1^{\rm ret}\right)}{q^2\left(t_1^{\rm ret},t_2\right)\left(1+v_1\left(t_1^{\rm ret}\right)\right)}\right.$$
$$\left. +(t-t_2)\frac{1-v_2(t_2)}{q^2\left(t_1^{\rm ret},t_2\right)\left(1+v_1\left(t_1^{\rm ret}\right)\right)} + \frac{1+v_2(t_2)}{q\left(t_1^{\rm ret},t_2\right)\left(1+v_1\left(t_1^{\rm ret}\right)\right)}\right]$$
$$+\int_{t_2^{\rm ret}(t)}^{t}\mathrm{d}t_2\left[-z_2(t_2)v_2(t_2)\frac{1-v_1(t_1')}{q^2(t_1',t_2)(1+v_1(t_1'))}\right.$$
$$\left.\left. -(t-t_2)\frac{1+v_2(t_2)}{q^2(t_1',t_2)(1+v_1(t_1'))} + \frac{1-v_2(t_2)}{q(t_1',t_2)(1+v_1(t_1'))}\right]\right\}. \qquad \text{(B.5)}$$

If we take the integrand in the form (B.4), we arrive at

$$\int_{\Sigma_t}\mathrm{d}\sigma_0\, Z\frac{e_1e_2}{(r_1r_2)^3}r^2b_1b_1 = \frac{e_1e_2}{2}\left\{-\int_{-\infty}^{t}\mathrm{d}t_1\left[z_1(t_1)v_1(t_1)\frac{1+v_2\left(t_2^{\rm ret}\right)}{q^2\left(t_1,t_2^{\rm ret}\right)\left(1-v_2\left(t_2^{\rm ret}\right)\right)}\right.\right.$$
$$\left. +(t-t_1)\frac{1+v_1(t_1)}{q^2\left(t_1,t_2^{\rm ret}\right)\left(1-v_2\left(t_2^{\rm ret}\right)\right)} + \frac{1-v_1(t_1)}{q\left(t_1,t_2^{\rm ret}\right)\left(1-v_2\left(t_2^{\rm ret}\right)\right)}\right]$$
$$+\int_{-\infty}^{t}\mathrm{d}t_2\left[z_2(t_2)v_1\left(t_1^{\rm ret}\right)\frac{1-v_2(t_2)}{q^2\left(t_1^{\rm ret},t_2\right)\left(1+v_1\left(t_1^{\rm ret}\right)\right)}\right.$$

$$
+ (t-t_2)\frac{1-v_1\left(t_1^{\rm ret}\right)}{q^2\left(t_1^{\rm ret},t_2\right)\left(1+v_1\left(t_1^{\rm ret}\right)\right)} - \frac{v_1\left(t_1^{\rm ret}\right)(1+v_2(t_2))}{q\left(t_1^{\rm ret},t_2\right)\left(1+v_1\left(t_1^{\rm ret}\right)\right)}\Bigg]
$$

$$
+ \int_{t_2^{\rm ret}(t)}^t \mathrm{d}t_2 \left[z_2(t_2)v_1(t_1')\frac{1+v_2(t_2)}{q^2(t_1',t_2)(1+v_1(t_1'))} \right.
$$

$$
\left.\left. - (t-t_2)\frac{1-v_1(t_1')}{q^2(t_1',t_2)(1+v_1(t_1'))} - \frac{v_1(t_1')(1-v_2(t_2))}{q(t_1',t_2)(1+v_1(t_1'))}\right]\right\}. \tag{B.6}
$$

Subtracting (B.6) from (B.5), we obtain the identity

$$
\begin{aligned}
&-\int_{-\infty}^t \mathrm{d}t_1 \frac{\mathrm{d}}{\mathrm{d}t_1}\frac{t-t_1+z_1(t_1)}{z_1(t_1)-z_2\left(t_2^{\rm ret}\right)} - \int_{-\infty}^t \mathrm{d}t_2 \frac{\mathrm{d}}{\mathrm{d}t_2}\frac{t-t_2-z_2(t_2)}{z_1\left(t_1^{\rm ret}\right)-z_2(t_2)} - \int_{t_2^{\rm ret}(t)}^t \mathrm{d}t_2 \frac{\mathrm{d}}{\mathrm{d}t_2}\frac{t-t_2+z_2(t_2)}{z_1(t_1')-z_2(t_2)}\\
&\quad = -\frac{z_1(t)}{z_1(t)-z_2\left[t_2^{\rm ret}(t)\right]} + \frac{z_2(t)}{z_1\left[t_2^{\rm ret}(t)\right]-z_2(t)}\\
&\qquad + \frac{z_2\left[t_2^{\rm ret}(t)\right]+t-t_2^{\rm ret}(t)}{z_1(t)-z_2\left[t_2^{\rm ret}(t)\right]} - \frac{z_2(t)}{z_1\left[t_2^{\rm ret}(t)\right]-z_2(t)} = 0.
\end{aligned}
$$

The remaining terms of equation (B.2) constitute the polynomial being the following sum of partial derivatives:

$$
\begin{aligned}
&\frac{e_1e_2}{4\pi}\frac{Z}{qr_1^2r_2^2}\left\{a_1a_2\left[\frac{1}{2}(t-t_1)^2+\frac{1}{2}(t-t_2)^2-\frac{q^2}{2}\right]-(t-t_1)(Z-z_2)b_1a_2\right.\\
&\qquad\left. -(Z-z_1)(t-t_2)a_1b_2+(t-t_1)(t-t_2)b_1b_2\right\}\\
&\quad = \frac{e_1e_2}{4\pi}\left\{\frac{\partial}{\partial t_1}\left[\frac{r^2}{q^2r_1}+\frac{r_2}{q^2}k_1k_1-\frac{t-t_1}{q^2}+ZG_1\right]\right.\\
&\qquad\left. +\frac{\partial}{\partial t_2}\left[-\frac{r^2}{q^2r_2}-\frac{r_1}{q^2}k_1k_1+\frac{t-t_2}{q^2}+ZG_2\right]+\frac{\partial^2}{\partial t_1\partial t_2}ZG\right\}
\end{aligned} \tag{B.7}
$$

where the functions G_1, G_2 and G are defined in equation (A.6).

The integration of the first-order partial derivatives gives

$$
\begin{aligned}
&\frac{e_1e_2}{2}\left\{\int_{-\infty}^t \mathrm{d}t_1\left[-\frac{v_1(t_1)}{q\left(t_1,t_2^{\rm ret}\right)}-(t-t_1)\frac{v_1(t_1)+v_2\left(t_2^{\rm ret}\right)-2v_1(t_1)v_2\left(t_2^{\rm ret}\right)}{q^2\left(t_1,t_2^{\rm ret}\right)\left(1-v_2\left(t_2^{\rm ret}\right)\right)}\right.\right.\\
&\qquad\left. +\left[z_1(t_1)+t-t_1\right]\mathcal{A}\left(t_1,t_2^{\rm ret}\right)\right]+\int_{-\infty}^t \mathrm{d}t_2\left[-\frac{v_2(t_2)}{q\left(t_1^{\rm ret},t_2\right)}\right.\\
&\qquad -(t-t_2)\frac{v_1\left(t_1^{\rm ret}\right)+v_2(t_2)+2v_1\left(t_1^{\rm ret}\right)v_2(t_2)}{q^2\left(t_1^{\rm ret},t_2\right)\left(1+v_1\left(t_1^{\rm ret}\right)\right)}\\
&\qquad\left. +[z_2(t_2)-t+t_2]\mathcal{B}\left(t_1^{\rm ret},t_2\right)\right]+\int_{t_2^{\rm ret}(t)}^t \mathrm{d}t_2\left[-\frac{v_2(t_2)}{q(t_1',t_2)}\right.\\
&\qquad\left.\left. -(t-t_2)\frac{v_1(t_1')-v_2(t_2)-2v_1(t_1')v_2(t_2)}{q^2(t_1',t_2)(1+v_1(t_1'))}+[z_2(t_2)+t-t_2]\mathcal{C}(t_1',t_2)\right]\right\}
\end{aligned} \tag{B.8}
$$

where the expressions $\mathcal{A}\big(t_1, t_2^{\text{ret}}\big)$, $\mathcal{B}\big(t_1^{\text{ret}}, t_2\big)$ and $\mathcal{C}(t_1', t_2)$ are defined in appendix A (see equations (A.12)).

The double derivative term can be expressed in the form

$$\frac{\partial}{\partial t_1}\left[Z\frac{\partial}{\partial t_2}\left(\frac{r^2}{qr_1r_2}+\frac{k_1k_2}{q}\right)-\frac{r_2}{q^2}k_1k_2\right] \tag{B.9}$$

or

$$\frac{\partial}{\partial t_2}\left[Z\frac{\partial}{\partial t_1}\left(\frac{r^2}{qr_1r_2}+\frac{k_1k_2}{q}\right)+\frac{r_1}{q^2}k_1k_2\right]. \tag{B.10}$$

To integrate the expression partial derivative in t_2 we apply the integration rule (5.4). We obtain

$$\begin{aligned}\frac{e_1e_2}{2}\Bigg\{&\int_{-\infty}^{t}\mathrm{d}t_1\left[-(t-t_1)\frac{1-v_1(t_1)}{q^2\big(t_1,t_2^{\text{ret}}\big)}-\frac{1-v_1(t_1)}{q\big(t_1,t_2^{\text{ret}}\big)}+[z_1(t_1)+t-t_1]\mathcal{D}_{01}\big(t_1,t_2^{\text{ret}}\big)\right]\\&+\int_{-\infty}^{t}\mathrm{d}t_2\left[(t-t_2)\frac{1+v_2(t_2)}{q^2\big(t_1^{\text{ret}},t_2\big)}+[z_2(t_2)-t+t_2]\mathcal{E}_{01}\big(t_1^{\text{ret}},t_2\big)\right]\\&+\int_{t_2^{\text{ret}}(t)}^{t}\mathrm{d}t_2\left[(t-t_2)\frac{1-v_2(t_2)}{q^2(t_1',t_2)}+[z_2(t_2)+t-t_2]\mathcal{F}_{01}(t_1',t_2)\right]\Bigg\}\end{aligned} \tag{B.11}$$

where

$$\begin{aligned}\mathcal{D}_{01}\big(t_1,t_2^{\text{ret}}\big)&=\mathcal{D}_0\big(t_1,t_2^{\text{ret}}\big)+\mathcal{D}_1\big(t_1,t_2^{\text{ret}}\big)\\\mathcal{E}_{01}\big(t_1^{\text{ret}},t_2\big)&=\mathcal{E}_0\big(t_1^{\text{ret}},t_2\big)+\mathcal{E}_1\big(t_1^{\text{ret}},t_2\big)\\\mathcal{F}_{01}(t_1',t_2)&=\mathcal{F}_0(t_1',t_2)+\mathcal{F}_1(t_1',t_2)\end{aligned}$$

and the functions in right-hand sides are given by equations (A.16) and (A.17).

The integration due to the coordinate system centred on the world line of the second particle leads to

$$\begin{aligned}\frac{e_1e_2}{2}\Bigg\{&\int_{-\infty}^{t}\mathrm{d}t_1\left[-(t-t_1)\frac{1-v_1(t_1)}{q^2\big(t_1,t_2^{\text{ret}}\big)}+[z_1(t_1)+t-t_1]\mathcal{D}_{02}\big(t_1,t_2^{\text{ret}}\big)\right]\\&+\int_{-\infty}^{t}\mathrm{d}t_2\left[(t-t_2)\frac{1+v_2(t_2)}{q^2\big(t_1^{\text{ret}},t_2\big)}+\frac{1+v_2(t_2)}{q\big(t_1^{\text{ret}},t_2\big)}+[z_2(t_2)-t+t_2]\mathcal{E}_{02}\big(t_1^{\text{ret}},t_2\big)\right]\\&+\int_{t_2^{\text{ret}}(t)}^{t}\mathrm{d}t_2\left[(t-t_2)\frac{1-v_2(t_2)}{q^2(t_1',t_2)}-\frac{1-v_2(t_2)}{q(t_1',t_2)}+[z_2(t_2)+t-t_2]\,\mathcal{F}_{02}(t_1',t_2)\right]\Bigg\}\end{aligned} \tag{B.12}$$

where

$$\begin{aligned}\mathcal{D}_{02}\big(t_1,t_2^{\text{ret}}\big)&=\mathcal{D}_0\big(t_1,t_2^{\text{ret}}\big)+\mathcal{D}_2\big(t_1,t_2^{\text{ret}}\big)\\\mathcal{E}_{02}\big(t_1^{\text{ret}},t_2\big)&=\mathcal{E}_0\big(t_1^{\text{ret}},t_2\big)+\mathcal{E}_2\big(t_1^{\text{ret}},t_2\big)\\\mathcal{F}_{02}(t_1',t_2)&=\mathcal{F}_0(t_1',t_2)+\mathcal{F}_2(t_1',t_2).\end{aligned}$$

The functions on the right-hand sides are given by equations (A.16) and (A.18).

Summing up all the contributions (B.8), (B.11) and either (B.5) or (B.6) we obtain the expression

$$\int_{\Sigma_t} \mathrm{d}\sigma_0 Z T^{00}_{\mathrm{int}} = -e_1 e_2 \int_{-\infty}^{t} \mathrm{d}t_1 [z_1(t_1) v_1(t_1) + t - t_1] \frac{1 + v_2\left(t_2^{\mathrm{ret}}\right)}{q^2\left(t_1, t_2^{\mathrm{ret}}\right)\left(1 - v_2\left(t_2^{\mathrm{ret}}\right)\right)}$$
$$+ e_1 e_2 \int_{-\infty}^{t} \mathrm{d}t_2 [z_2(t_2) v_2(t_2) + t - t_2] \frac{1 - v_1\left(t_1^{\mathrm{ret}}\right)}{q^2\left(t_1^{\mathrm{ret}}, t_2\right)\left(1 + v_1\left(t_1^{\mathrm{ret}}\right)\right)}$$
$$- z_1(t) \frac{e_1 e_2}{z_1(t) - z_2\left[t_2^{\mathrm{ret}}(t)\right]}.$$

Taking into account the first term on the right-hand side of (B.1), we obtain the stable expression (6.5) for the interference part of the third component of the 'centre-of-mass' conserved quantity plus changeable term (6.6). If we use (B.12) instead of (B.11), we obtain the changeable term

$$-z_2(t) \frac{e_1 e_2}{z_1\left[t_1^{\mathrm{ret}}(t)\right] - z_2(t)}$$

which should be substituted for the last term in this expression.

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